# Quantum computation

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# introduction: Divincenzo criterion

After introducing some quantum algorithm and quantum computation model, We now seriously to make it true. We need to find a proper physical system to realize the quantum computation. There are several criterions for a proper physical system

- there is a local system can be used to present qubit (two level system is a good approximation)
- the whole system can be prepared in a initial state (can be locally cooled down to the ground state), such as $|+\rangle^{\bigotimes n}$
- the whole gates in a universal set can be implemented, such as $CZ$ and local unitary transformation
- the outcome can be measured efficiently
- the coherence time of the whole system is long enough to complete $10^4$ operating (depend on $\tau_Q/\tau_{op}$)
- the different qubits is distinguishable
- the system is scalable

# introduction: quantum computation system

based on the former criterion, there are several systems are proposed to realize quantum computation.

- linear optics system:
  - polarization of photon can be used as qubit; long coherence time; hybrid to quantum communication
  - probabilistic two qubits gate; bad scalability
- ion trap system
  - local ion inner state can be used as qubit; mediate coherence time; efficient output;good gate operation
  - mediate scalability
- Josephson junction system
- NMR system
- quantum dot system
- Quantum Hall system

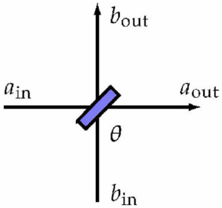
# elementary optical component

the basic particle in electromagnetic field is photon, their are several elementary optical components in linear optics

- single-model shift:It changes the phase of the electromagnetic field in a given mode as

$$\hat{a}_{out}^{\dagger} = e^{i\phi\hat{a}_{in}^{\dagger}\hat{a}_{in}}\hat{a}_{in}^{\dagger}e^{-i\phi\hat{a}_{in}^{\dagger}\hat{a}_{in}} = e^{i\phi}\hat{a}_{in}^{\dagger} \qquad (1)$$

  the corresponding Hamiltonian is $H_{\phi} = \phi\hat{a}_{in}^{\dagger}\hat{a}_{in}$
- beam splitter: it consists of a semireflective mirror, the light on this mirror, part will be reflected and part will be transmitted.



# elementary optical component

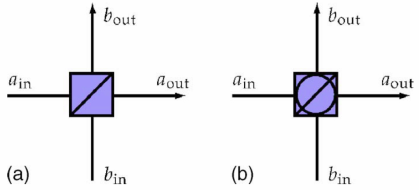
- the $BS$ evolution in operator form is

$$\begin{aligned}\hat{a}^\dagger_{out} &= \cos\theta \hat{a}^\dagger_{in} + ie^{-i\varphi}\sin\theta \hat{b}^\dagger_{in} \\ \hat{b}^\dagger_{out} &= ie^{i\varphi}\sin\theta \hat{a}^\dagger_{in} + \cos\theta \hat{b}^\dagger_{in}\end{aligned} \quad (2)$$

  the corresponding Hamiltonian is

$$H_{BS} = \theta e^{i\varphi}\hat{a}^\dagger_{in}\hat{b}_{in} + \theta e^{-i\varphi}\hat{a}_{in}\hat{b}^\dagger_{in} \quad (3)$$

- Polarizing Beam Splitter: the splitter involved the polarization of the photon.



# elementary optical component

- the relation of outgoing mode and the incoming mode

$$\hat{a}^{\dagger}_{in,H} \rightarrow \hat{a}^{\dagger}_{out,H} \qquad \text{and} \qquad \hat{a}^{\dagger}_{in,V} \rightarrow \hat{b}^{\dagger}_{out,V} \tag{4}$$

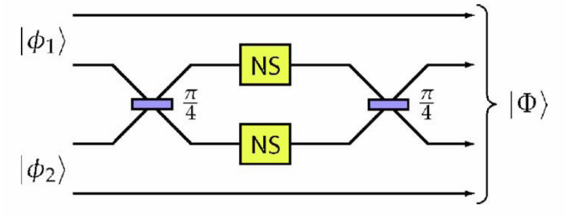
$$\hat{b}^{\dagger}_{in,H} \rightarrow \hat{b}^{\dagger}_{out,H} \qquad \text{and} \qquad \hat{b}^{\dagger}_{in,V} \rightarrow \hat{a}^{\dagger}_{out,V} \tag{5}$$

  Due to the linear character,(or no interaction between photoes) the two qubits gate can not be deterministic realized, it can only be probabilistic constructed.

- Hong-ou-Mandel effect: when two photons in separate spatial modes interacting on a $50:50$ beam splitter, the bosonic nature of the electromagnetic field gives rise to photon bunching: the incoming photons pair off together.

# two-qubit gate

- the two-qubit CZ gate can be realized as the following




where there are Hong-Ou-Mandel interference and two box $NS$. The effect of box $NS$ like this

$$\alpha|0\rangle + \beta|1\rangle + |\gamma|2\rangle \rightarrow \alpha|0\rangle + \beta|1\rangle - |\gamma|2\rangle \tag{6}$$

# two-qubit gate

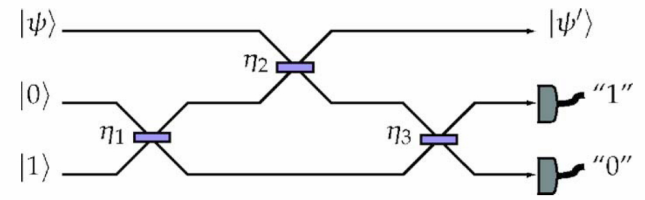
- suppose the input state is $|\phi_1\rangle|\phi_2\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle)$
- the first Hong-Ou-Mandel interference just operate on the state $|1,1\rangle$, to create state $|2,0\rangle$ and $|0,2\rangle$
- two $NS$ boxes add minus to these two state respectively.
- The second Hong-Ou-Mandel turn $|2,0\rangle$ and $|0,2\rangle$ back to $|1,1\rangle$
- after the circuit the state will be

$$|\Phi\rangle = \alpha\gamma|0,0\rangle + \alpha\delta|0,1\rangle + \beta\gamma|1,0\rangle - \beta\delta|1,1\rangle$$

  this is no longer a separate state
- choose $\alpha = \beta = \gamma = \delta = 1/\sqrt{2}$ the state is a maximally entangled state, the success probability is square of the success probability of $NS$

# realize $NS$ Box

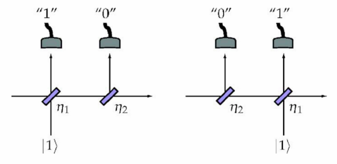
- Now the key problem to realize CZ gate turn to realize $NZ$ Box, the first one is KLM proposal




which is a three-port device with two ancillary mode, The parameters $\eta_1 = \eta_3 = 1/(4 - 2\sqrt{2})$ and $\eta_2 = 3 - 2\sqrt{2}$. For any input $\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$,this Box success with probability $1/4$ when the detector $D_1$ and $D_2$ detect zero and one photon.

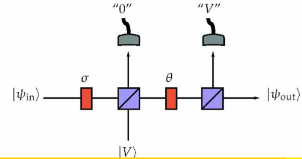
# realize $NS$ Box

- a simplified proposal is Ralph and White's proposal




this scheme is just need two beam splitters, and the success probability is $(3-\sqrt{2})/7$

- There is a polarization version of RW proposal which change beam splitter by polarization rotations. This proposal is more convenient for experiment



# realize $NS$ Box

- suppose the input mode is $\hat{a}_H$

$$
\begin{aligned}
&(\alpha + \beta\hat{a}_H^\dagger + \tfrac{\gamma}{\sqrt{2}}\hat{a}_H^{\dagger 2})\hat{b}_V^\dagger \\
&\rightarrow [\alpha + \beta\cos\sigma\hat{a}_H^\dagger + \beta\sin\sigma\hat{a}_V^\dagger \\
&+\gamma/\sqrt{2}(\cos\sigma^2\hat{a}_H^{\dagger 2} + \sin 2\sigma\hat{a}_H^\dagger\hat{a}_V^\dagger + \sin^2\sigma\hat{a}_V^{\dagger 2})]\hat{b}_V^\dagger \\
&\rightarrow [\alpha + \beta\cos\sigma\hat{a}_H^\dagger + \tfrac{\gamma}{\sqrt{2}}\cos\sigma^2\hat{a}_H^{\dagger 2}]\hat{b}_V^\dagger \\
&\rightarrow [\alpha + \beta\cos\sigma(\cos\theta\hat{a}_H^\dagger + \sin\theta\hat{a}_V^\dagger) + \tfrac{\gamma}{\sqrt{2}}\cos\sigma^2(\cos\theta\hat{a}_H^\dagger + \sin\theta\hat{a}_V^\dagger)^2] \\
&(\text{-}\sin\theta\hat{a}_H^\dagger + \cos\theta\hat{a}_V^\dagger) \\
&\rightarrow [\alpha\cos\sigma|0\rangle + \beta\cos\sigma\cos 2\theta|1\rangle + \gamma\cos\sigma^2\cos\theta(1-\sin^2 3\theta)|2\rangle
\end{aligned}
$$

- let $\sigma = 150.5^o$ and $\theta = 61.5^o$, this scheme will yield the $NS$ gate with success probability $(3-\sqrt{2})/7$.

# to scalable quantum computer

- Though the realize of universal gates, the direct quantum computation is still impossible due to the bad scalability of the probabilistic two-qubit gate. The success probability will exponentially decay with the number of two-qubit gates.
- There are several methods to overcome this problem:
  - KLM protocal
  - Yoran-Reznik protocol
  - Nielsen protocol
  - Browne-Rudolph protocol
  - Duan protocol

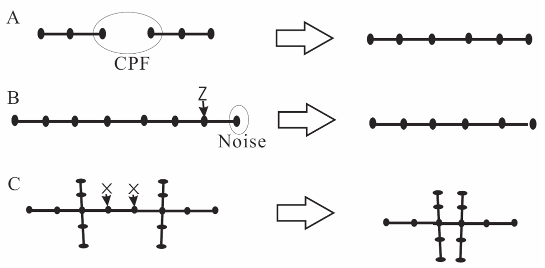
  the first one is based on teleportation trick to built a scalable quantum computer. The rest protocol are based on one-way computer of cluster state.

At first, we recall some properties of graph states, these properties make key roles in the protocol

- If we have two chains of cluster states each with $n$ qubits, we can join them to form a $1D$ cluster state of $2n$ qubits by successfully applying a $CZ$ gate on the end qubits of the two chains
- If we destroy the state of an end qubit of an $n$-qubit cluster chain, for instance, through an unsuccessful attempt of the $CZ$ gate, we can remove this bad qubit by performing a $Z$ measurement on its neighboring qubit, and recover a cluster state of $n-2$ qubits.
- We can shrink a cluster state by performing X measurements on all the connecting qubits

A character of the linear optical $CZ$ gate is critical: we can determine that the gate success or not

# Duan protocol




Now we use these properties to construct $1D$ cluster state

- If we have two sufficiently long cluster chains each with $n_0$ qubits, we connect them through a probabilistic $CZ$ gate. If it is fails, we can delete these destroyed qubit by $Z$ measurement. And we get two $n_0 - 2$ qubit chains.
- repeat

# Duan protocol

By the former method, the average number of connected chain is

$$\bar{n_1} = \sum_{i=0}^{n_0/2} 2(n_0 - 2i)p(1-p)^i \simeq 2n_0 - 4(1-p)/p \tag{7}$$

where the approximation made under the condition $e^{-n_0 p/2} \leq 1$. This equation set a critical value $n_c = 4(1-p)/p$, when $n_0 > n_c$ the number of chain qubit will increase.

- For the $rth(r \geq 1)$ round of successful connection, the chain length $n_r$, the total preparation time $T - r$, and the total number of attempts $M_r$ scale up respectively by the recursion relations $n_r = 2n_{r-1} - n_c, T_r = T_{r-1} + t_a/p, M_r = 2M_{r-1} + 1/p$ where $t_a$ denotes the time for each attempt of the $CZ$ gate.

# Duan protocol

$$T(n) = T_0 + (t_a/p)\log_2[(n - n_c)/(n_0 - n_c)] \quad (8)$$

$$M(n) = (M_0 + 1/p)(n - n_c)/(n_0 - n_c) - 1/p \quad (9)$$

where $T_0$ and $M_0$ are time and attempts of the probabilistic gates to prepare cluster chains with $n_0$ qubits respectively.

- Now we turn to construct a cluster chain with $n_0 > n_c$ qubits,The chain with $n_0 > n_c$ qubits can be generated by repeater protocol.
  - divide the task into $m = log_2 n$ steps
  - For the $ith$ step, building a $2^i - bit$ cluster state by connecting two $2^{i-1} - bit$ cluster through probabilistic a $CZ$ gate.
  - if it is failed, repeat from the beginning.

# Duan protocol

- The scaling of this preparation can be obtained as following
  - For the $ith$ step, the recursion relations of preparation time $T_i$ and attempts $M_i$ is

$$T_i = (1/p)(T_{i-1} + t_a) \quad (10)$$

$$M_i = (1/p)(2M_{i-1} + 1) \quad (11)$$

    with initial condition $T_1 = t_a/p$ and $M_1 = 1/p$

  - so get the scaling

$$T(n) \simeq t_a(1/p)^{\log_2 n} \quad (12)$$

$$M(n) \simeq (2/p)^{\log_2 n}/2 \quad (13)$$

# Duan protocol

- connecting the former two section results to get the final scaling of preparation cluster chain with $n$ qubits

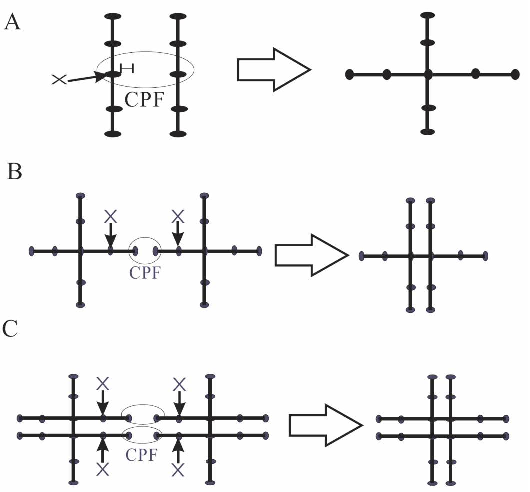
$$T(n) \simeq t_a(1/p)^{\log_2 n_c+1} + (t_a/p)log_2(n-n_c) \quad (14)$$

$$M(n) \simeq (2/p)^{\log_2 (n_c+1)}(n-n_c)/2 \quad (15)$$

To realize quantum computation, we need to construct a 2D cluster state from 1D cluster chains

- First, We construct some '$+$' shape state with long enough tails
- connect different '$+$' shapes by $CZ$ gate and $X$ measurements
- connect these centers qubits to form a complex lattice

# Duan protocol



# Duan protocol

- How many legs in a '+' shape is enough? For a $n_l$ qubits leg, we can do $n_l/2$ times probabilistic $CZ$ gates and the success probability is $p_c = 1-(1-p)^{n_l/2}$. To form a square lattice with N qubits, we need $2N$ connection operations and the success probability is $p_c^{2N}$. If we require the probability is sufficient large $1-\varepsilon$, then $n_l \approx (2/p)\ln(2N/\varepsilon)$.
- Finally, the scaling to preparation the 2D cluster state is

$$T(n) \simeq t_a(1/p)^{\log_2 4/p-3} + \frac{t_a}{p}log_2(\frac{4}{p}[\ln(2N/\varepsilon)-1]) + \frac{t_a}{p}ln(2N/\varepsilon)$$

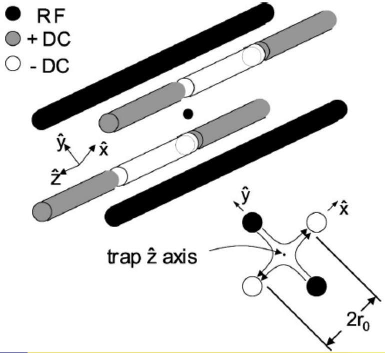
$$M(n) \simeq (2/p)^{2+\log_2(4/p-3)}N[ln(2N/\varepsilon)-1] + 2N/p$$

# Duan protocol

- arxiv: 0502120
- RMP 79,135 (2007)

# introduction pauling trap

- Since there is no minimal point in electromagnetic field, there is no way to confine a charged ion in static field
- using the Rf to make the potential quickly rotation, then can trap the ion.



# introduction pauling trap

- the static potential in this setup is given by

$$\Phi_d c = \kappa U_0[z^2 - (x^2 + y^2)]/2 \quad (16)$$

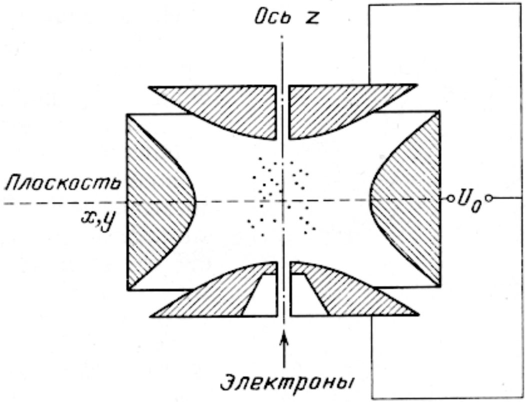
- the radiofrequency potential on the other two poles is

$$\Phi_{rf} = (V_0 cos\Omega_T t + U_r)(1 + (x^2 - y^2)/R^2)/2 \quad (17)$$

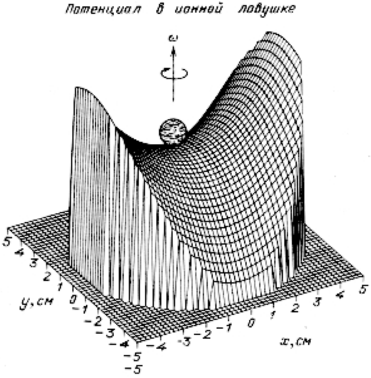
- The combination of this to potential, on average, will be a harmonic trap and $\omega_x, \omega_y \geq \omega_z$ typically.
- the hamiltonian of ions trapped in the trap is

$$H = \sum_{i=1}^{N} \frac{M}{2}(\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2 + \frac{|\vec{p_i}|^2}{M^2}) + \sum_{i=1,j>i}^{N} \frac{e^2}{4\pi\epsilon_0|\vec{r_i} - \vec{r_j}|}$$

# introduction pauling trap




а)




б)

# cooling in trap

To realize quantum computation, we need use the vibrational mode (phonon), and the system satisfy some conditions

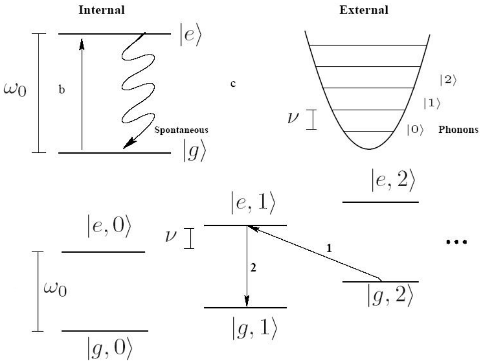
- the system is isolated enough to make the state of the system stable.
- we must cooling the system to low temperature, such that the former harmonic approximation is good and the ions in the ground state
    - doppler cooling: limit $k_B T \approx \hbar\Gamma/2$
    - sideband cooling

  then we can reach $k_B T \ll \hbar\omega_z$
- Lamb-Dicke criterion should be satisfied:the width of the ion oscillation should be samll compared to incident light wavelength

# cooling in trap

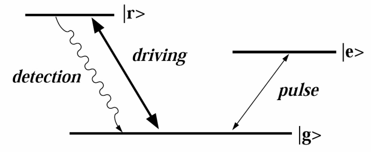
that is

$$\eta = 2\pi z_0/\lambda = \frac{2\pi\sqrt{\hbar/2NM\omega}}{\lambda} \tag{18}$$



# quantum computation

- qubit: two internal states of a cooling ion
- bus: phonon
- preparation a initial state: Doppler cooling and sideband cooling
- universal gates:
  - one-qubit unitary: an ion interact with laser described by J-C model
  - two-qubit gate: two ions interact by phonon
- measurement: Using laser to pump ions on the ground state to an excited state and observe the spontaneous emission which can be detected with almost 100%.



# ions dynamics

- there are $N$ ions in the trap,since $\omega_x, \omega_y \geq \omega_z$, the trap on the $x$ and $y$ direction are much stronger than $z$ direction. we suppose the ions can only move on the $z$ direction, then the the Hamiltonian is reduced to one dimensional situation

$$H_0 = \sum_{i=1}^{N} \frac{|\vec{p_i}|^2}{2M})$$

$$V = \sum_{i=1}^{N} \frac{M}{2}\omega_z^2 z_i^2 + \sum_{i=1,j>i}^{N} \frac{e^2}{4\pi\epsilon_0|\vec{r_i} - \vec{r_j}|}$$

$$L = T - V$$

similar as the lattice dynamics in solid physics, the ions will stay at the equilibration point and vibrate near that point

# ions dynamics

- we can approximate the potential function $V$ near the equilibration point as

$$
\begin{aligned}
&V(\bar{z_1}+q_1(t),\bar{z_2}+q_2(t),...,\bar{z_N}+q_N(t),)\\
=&V(\bar{z_1},\bar{z_2},...,\bar{z_N})+\sum_{m=1}^{N}\frac{\partial V}{\partial z_m(t)}|_{q_m=0}q_m(t)\\
+&\frac{1}{2}\sum_{n,m=1}^{N}\frac{\partial^2 V}{\partial z_m(t)\partial z_n(t)}|_{q_m,q_n=0}q_m(t)q_n(t)+...
\end{aligned}
\tag{19}
$$

Since $\bar{z_1},\bar{z_2},...,\bar{z_N}$ is the equilibration point, $\frac{\partial V}{\partial z_m(t)}|_{q_m=0}=0$, then the $L$ can be expressed as

$$
L(\dot{q_m},q_m)=\frac{M}{2}\sum_{m=1}^{N}(\dot{q_m})^2-\frac{1}{2}\sum_{n,m=1}^{N}V_{nm}q_nq_m \tag{20}
$$

# ions dynamics

- where $V_{nm}$ is a Hessian matrix with definite positive character
- the dynamics of the trapped ion is governed by the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0 \qquad (21)$$

- let the solution of $q_k$ with the form $C_k e^{-i\nu t}$, and we get

$$|\omega_z^2 V_{kl} - \nu^2 \delta_{kl}| = 0 \qquad (22)$$

Generally, there are up to $N$ nonnegative solutions $\nu_k$ and the solution of $q_k$ has the following form

$$q_k = \sum_{\alpha=1}^{N} D_k^\alpha Q_\alpha(t) \qquad k = 1, 2, ..., N \qquad (23)$$

where $Q_\alpha(t) = C_\alpha e^{-i\nu_\alpha t}$ And $D_k^\alpha$ is the eigenvectors of matrix $V$ by

# ions dynamics

$$\sum_{k=1}^{N} V_{k\alpha} D_k^\alpha = \mu_\alpha D_l^\alpha \tag{24}$$

- insert the expression of $q_k(t)$ into the Lagrange to get

$$L = \frac{m}{2} \sum_{\alpha=1}^{N} (\dot{Q_\alpha}^2 - \nu_\alpha^2 Q_\alpha^2)$$

where $\nu_\alpha = \omega_z \sqrt{\mu_\alpha}$, $Q_\alpha$ and $\nu_\alpha$ are named normal modes and normal frequencies respectively

- if define the canonical conjugated to $Q_\alpha$ is $P_\alpha = m\dot{Q}_\alpha$, then

$$H = \frac{1}{2m} \sum_{\alpha=1}^{N} P_\alpha^2 + \frac{1}{2m} \sum_{\alpha=1}^{N} \nu^2 Q_\alpha^2$$

# ions dynamics

- introduce creation and annihilation operator by

$$Q_\alpha \to \hat{Q}_\alpha = \sqrt{\frac{\hbar}{2m\nu_\alpha}}(\hat{a_\alpha}^\dagger + \hat{a_\alpha})$$

$$P_\alpha \to \hat{P}_\alpha = i\sqrt{\frac{\hbar m\nu_\alpha}{2}}(\hat{a_\alpha}^\dagger - \hat{a_\alpha})$$

- then the quantized hamiltonian will be

$$\hat{H} = \sum_{\alpha=1}^{N} \hbar\nu_\alpha(\hat{a_\alpha}^\dagger \hat{a_\alpha} + \frac{1}{2}) \tag{25}$$

this is the quantum form of the ion trap system.

- the express of the position is

# laser-ion interaction

$$z_j(t) = \bar{z_j} + \sum_{\alpha=1}^{N} D_i^{\alpha} \sqrt{\frac{\hbar}{2m\nu_\alpha}} (\hat{a_\alpha}^\dagger + \hat{a_\alpha}) \quad (26)$$

$$= \bar{z_j} + \sum_{\alpha_1}^{N} K_i^{\alpha} (\hat{a_\alpha}^\dagger + \hat{a_\alpha}) \quad (27)$$

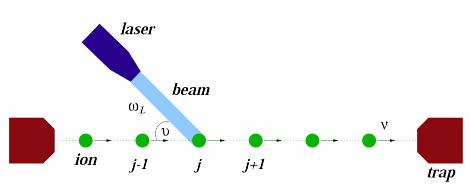
where $K_i^\alpha = D_i^\alpha/(\mu_\alpha)^1/4$ and $z_0 = \sqrt{\hbar}2m\omega_z$
Now we turn to implement the one-qubit unitary gate on this system, we use the laser to interact with a ion

- a laser with electromagnetic field

$$E = E_0\vec{\epsilon}\cos(\omega_L t - \vec{\kappa}.\vec{q} + \phi) \quad (28)$$

where $\vec{\epsilon}$ is a polarization vector

# laser-ion interaction




- the interaction between ion and electromagnetic is dipole inter action

$$V_j = -q_e \vec{r}_j . \vec{E}(t, \vec{R}_j)$$

- insert the expression of position into the interaction formula to get

$$V_j = -q_e[(\vec{r}_{eg})_j \hat{\sigma}_{+j} + (\vec{r}_{eg})^*_j \hat{\sigma}_{-j}] . \frac{E_0 \vec{\epsilon}}{2}[e^{-i[\omega_L t - \eta_j(\hat{a}^\dagger + \hat{a}) + \phi_j]} + H.C]$$

where $(\vec{r}_{eg})_j = \langle e_j | \vec{r}_j | g_j \rangle, \hat{\sigma}_{+j} = |e_j\rangle\langle g_j|$

# laser-ion interaction

- the free Hamiltonian of the $jth$ ion is

$$H_{0j} = \frac{\hbar\omega_0}{2} + \hbar\nu\hat{a}^\dagger\hat{a}$$

where we just consider the lowest mode of the phonon which is named $COM$ and $\nu = \omega_z$

- transform the whole Hamiltonian to interaction picture to get

$$\begin{aligned}\hat{H}_j &= \hat{U_0}^\dagger V_j \hat{U_0} \\ &= \frac{\hbar\lambda_j}{2}\hat{\sigma}_{+j}\mathrm{exp}[i\eta_j(\hat{a}^\dagger e^{i\nu t} + \hat{a}e^{-i\nu t})]e^{-i\delta t} + H.C.\end{aligned}$$

where $\lambda_j = -q_e E_0[(\vec{r}_{eg})_j.\vec{\epsilon}]e^{-i\phi_j}/\hbar$

- let the detuning $\delta = \omega_L - \omega_0 = k\nu$, then the final Hamiltonian can be

$$\hat{H}_j = \hbar\sum_n[\frac{\Omega_j^{n,k}}{2}(|e_j\rangle\langle g_j|\bigotimes|n+|k|\rangle\langle n|) + H.C]$$

# laser-ion interaction

- the corresponding unitary evolution operator is

$$\hat{U}_j^+ = \sum_n \cos(\frac{|\Omega_j^{n,k}|t}{2})[(|g_j\rangle\langle g_j| \bigotimes |n+|k|\rangle\langle n+|k||) + (|e_j\rangle\langle e_j| \bigotimes |n\rangle\langle n|)]$$
$$- i\sum_n \sin(\frac{|\Omega_j^{n,k}|t}{2})[(|g_j\rangle\langle e_j| \bigotimes |n+|k|\rangle\langle n|)e^{-i\bar{\phi}_j}$$
$$+ (|e_j\rangle\langle g_j| \bigotimes |n\rangle\langle n+|k||)e^{i\bar{\phi}_j}] + \sum_{n=0}^{|k|-1}(|e_j\rangle\langle e_j| \bigotimes |n\rangle\langle n|)$$

- where $\Omega_j^{n,k}$ is called Rabi frenquency. For control the time of evolution, we can
  - A $4\pi - pulse$ returns the system back to its initial state.
  - A $2\pi - pulse$ changes the sign of the state.
  - A $\pi - pulse$ imply change of the $|e_j\rangle$ and $|g_j\rangle$

# laser-ion interaction

- In quantum computation, we just interesting the case when $k = 0$ and $k = -1$, So the unitary evolution can simplified as

$$\hat{A}_j = \sum_n \cos(\frac{|A_j^n|t}{2})[(|e_j\rangle\langle e_j| \bigotimes |n\rangle\langle n|) + (|g_j\rangle\langle g_j| \bigotimes |n\rangle\langle n|)]$$
$$- i\sum_n \sin(\frac{|A_j^n|t}{2})[(|e_j\rangle\langle g_j| \bigotimes |n\rangle\langle n|)e^{-i\bar{\phi_j}}$$
$$+ (|g_j\rangle\langle e_j| \bigotimes |n\rangle\langle n|)e^{i\bar{\phi_j}}]$$

$$\hat{B}_j = \sum_n \cos(\frac{|A_j^n|t}{2})[(|e_j\rangle\langle e_j| \bigotimes |n\rangle\langle n|) + (|g_j\rangle\langle g_j| \bigotimes |n+1\rangle\langle n+1|)]$$
$$- i\sum_n \sin(\frac{|A_j^n|t}{2})[(|e_j\rangle\langle g_j| \bigotimes |n\rangle\langle n+1|)e^{-i\bar{\phi_j}}$$
$$+ (|g_j\rangle\langle e_j| \bigotimes |n+1\rangle\langle n|)e^{i\bar{\phi_j}}] + |g_j\rangle\langle g_j| \bigotimes |0\rangle\langle 0|$$

where $|A_j^n| = |\lambda_j|$ and$|B_j^n| = |\lambda_j|\eta_j\sqrt{n+1}$under the Lamd-Dicke limit.

# Lamd-Dicke limit

- The Lamb-Dicke limit corresponds physically to the situation where the spatial extent of the vibrational motion of the ion $z_0$ is much smaller than the wavelength $\Lambda$ of the laser, where $\eta_j \simeq \kappa z_0$ and $\kappa = 2\pi/\Lambda$
- We may rewrite the Lamb-Dicke parameter of the $j$ ion of $N$ ions in the COM mode to the form $\eta_j^2 = E_r/\hbar\omega_z$, where $E_r = \hbar^2/2mN$ is the recoil energy. It can be shown that the trapped ion emits spontaneously photons of the average energy $\hbar\omega - E_r$, where $_0$ is for the atomic frequency. Taking into account the Lamb-Dicke limit $(E_r \ll \hbar\omega_z)$ we may say that during the spontaneous emission the change in the vibrational state of the ion is very unlikely. In other words, the trapped ion in the Lamb-Dicke regime decays spontaneously mostly on the carrier.
- The Lamb-Dicke parameter for a single trapped ion equals to $\bar{\eta}$ and for an ion from the string of $N$ ions in the COM mode is given as $\eta_j = \bar{\eta}/\sqrt{N}$. It means that we can reach the Lamb-Dicke limit for N ions even if the limit is not fulfilled for single ions.

# one-qubit gate

- the one-qubit gate can be realized by the former unitary transformation when $k = 0$ by parameter $t = l\pi/|\lambda_j|$ and $\phi_j \to \phi_j + \pi/2$, that is

$$
\begin{aligned}
\hat{A}_j^l(\phi_j) = & \sum_n \cos(\frac{l\pi}{2})[(|e_j\rangle\langle e_j| \bigotimes |n\rangle\langle n|) + (|g_j\rangle\langle g_j| \bigotimes |n\rangle\langle n|)] \\
& + \sum_n \sin(\frac{l\pi}{2})[(|e_j\rangle\langle g_j| \bigotimes |n\rangle\langle n|)e^{-i\bar{\phi_j}} \\
& + (|g_j\rangle\langle e_j| \bigotimes |n\rangle\langle n|)e^{i\bar{\phi_j}}]
\end{aligned}
$$

this unitary corresponding to the rotation matrix

$$
\begin{pmatrix} \cos(\theta/2) & e^{i\phi}\sin(\theta/2) \\ -e^{-i\phi}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{29}
$$

# two-qubit gate

- to realize a two-qubit $CNOT$ gate, we need an auxiliary internal level $|r\rangle$
- using two laser with frequency $\omega_0^{eg} = (E_e - E_g)\hbar$ and $\omega_0^{rg} = (E_r - E_g)\hbar$, the evolution with $k = 1$ is given

$$
\begin{aligned}
\hat{B}_j^{l,I} &= \cos(\frac{l\pi}{2})[(|e_j\rangle\langle e_j| \bigotimes |0\rangle\langle 0|) + (|g_j\rangle\langle g_j| \bigotimes |1\rangle\langle 1|)] \\
&- i\sin(\frac{l\pi}{2})[(|e_j\rangle\langle g_j| \bigotimes |0\rangle\langle 1|)e^{-i\bar{\phi}_j} \\
&+ (|g_j\rangle\langle e_j| \bigotimes |1\rangle\langle 0|)e^{i\bar{\phi}_j}] + |g_j\rangle\langle g_j| \bigotimes |0\rangle\langle 0|
\end{aligned}
$$

$$
\begin{aligned}
\hat{B}_j^{l,II} &= \cos(\frac{l\pi}{2})[(|r_j\rangle\langle r_j| \bigotimes |0\rangle\langle 0|) + (|g_j\rangle\langle g_j| \bigotimes |1\rangle\langle 1|)] \\
&- i\sin(\frac{l\pi}{2})[(|r_j\rangle\langle g_j| \bigotimes |0\rangle\langle 1|)e^{-i\bar{\phi}_j} \\
&+ (|g_j\rangle\langle r_j| \bigotimes |1\rangle\langle 0|)e^{i\bar{\phi}_j}] + |g_j\rangle\langle g_j| \bigotimes |0\rangle\langle 0|
\end{aligned}
$$

where we only consider the case $n = 0$ and $n = 1$, we neglect the higher level.

# two-qubit gate

- the $CNOT$ can be realized by the following evolution operator sequence on qubit $m1$ and $m2$

$$\hat{A}^{1/2}_{m2}(\pi)\hat{B}^{1,I}_{m1}\hat{B}^{2,II}_{m2}\hat{B}^{1,I}_{m1}\hat{A}^{1/2}_{m2}(0)$$

  where $\hat{A}^{l}_{j}$ defined in the one-qubit gate case

- $\hat{B}^{1,I}_{m1}\hat{B}^{2,II}_{m2}\hat{B}^{1,I}_{m1}$ operate as a $CZ$ gate

$$\begin{array}{ccccccc}
 & \hat{\mathcal{B}}^{1,I}_{m_1} & & \hat{\mathcal{B}}^{2,II}_{m_2} & & \hat{\mathcal{B}}^{1,I}_{m_1} & \\
|g_{m_1}\rangle|g_{m_2}\rangle|0\rangle & \longrightarrow & |g_{m_1}\rangle|g_{m_2}\rangle|0\rangle & \longrightarrow & |g_{m_1}\rangle|g_{m_2}\rangle|0\rangle & \longrightarrow & |g_{m_1}\rangle|g_{m_2}\rangle|0\rangle \\
|g_{m_1}\rangle|e_{m_2}\rangle|0\rangle & \longrightarrow & |g_{m_1}\rangle|e_{m_2}\rangle|0\rangle & \longrightarrow & |g_{m_1}\rangle|e_{m_2}\rangle|0\rangle & \longrightarrow & |g_{m_1}\rangle|e_{m_2}\rangle|0\rangle \\
|e_{m_1}\rangle|g_{m_2}\rangle|0\rangle & \longrightarrow & -i|g_{m_1}\rangle|g_{m_2}\rangle|1\rangle & \longrightarrow & i|g_{m_1}\rangle|g_{m_2}\rangle|1\rangle & \longrightarrow & |e_{m_1}\rangle|g_{m_2}\rangle|0\rangle \\
|e_{m_1}\rangle|e_{m_2}\rangle|0\rangle & \longrightarrow & -i|g_{m_1}\rangle|e_{m_2}\rangle|1\rangle & \longrightarrow & -i|g_{m_1}\rangle|e_{m_2}\rangle|1\rangle & \longrightarrow & -|e_{m_1}\rangle|e_{m_2}\rangle|0\rangle
\end{array}$$

- finally, the $CNOT$ be realized by local unitary transformation $\hat{A}^{l}_{j}$s.

# decoherence in ion trap

- motional state decoherence
  - instability of trap parameters
  - We also have to count on (i) the micromotion, (ii) the Coulomb repulsion between the ions making the motional modes (except the COM mode) anharmonic in reality and (iii) stray electrode fields causing possible excitations of the ion motion
  - We have considered just a single motional mode in our approach, but there are also other 3N . 1 modes present and the cross-coupling between the modes appears.
- inelastic and elastic collisions with the background gas, even though experiments are carried out in an excellent environment $(p \simeq 10^{-8} Pa)$

# decoherence in ion trap

- Internal state decoherence, The type of the decoherence discussed here can be eliminated by a proper choice of metastable excited states with long lifetimes.
- Operational decoherence
  - the pulse duration and the phase adjustment is not exactly
  - Due to the laser spatial intensity profile, there is a probability (if the ions are spaced too closely) that the state of a neighbouring ion will be affected
  - off-resonant transitions are always present and we have to control the laser power very carefully to avoid their excitations.

# decoherence in ion trap


- arxiv: 9608011.
- arxiv: 011204.
- arxiv: 0809.4368.
- RMP 75, 281 (2003).
- RMP 82, 2609 (2010).
- PRL 74, 4091 (1995).

# introduction

- there are two effects of the environment for a quantum system
  - dissipation of the energy which denotes by $T_1$
  - decoherence which denotes by $T_2$

  Generally, $T_2 \ll T_1$. But for some special decoherece mechanics this will not be true.
- the main difficulty on the road to quantum computation is decoherence
- decoherence plays a key role in connection quantum world and classical world
- the decoherence will turn a superposition state to a ensemble, that is, make the quantum computation to classical computation.

# decoherence

- For a general quantum state, it can be expressed as a density matrix,

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \cdots & \cdots & \cdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}$$

- the non diagonal elements $\rho_{ij}, i \neq j$ describe the coherent. Decoherence will make the non diagonal element disappear, and the character time is $T_1$. while the character disappear time of the diagonal term is named $T_2$
- For a multiqubit system, the decoherence of different qubits may be cooperate or just the same.
- the theoretic tools for decoherence is master equation and lindblad equation.

# decoherence

the decoherence is due to the environment, so we can transform overcome it by considering the whole system

- Operator-sum representation

$$\hat{\rho}(t) = \hat{U}(\hat{\rho}(t_0) \bigotimes |0\rangle\langle 0|)\hat{U}^\dagger$$

where $\hat{\rho}(t_0)$ is the initial state of the system $A$ and $|0\rangle$ is the initial state of system $B$, $U$ is the evolution operator of the whole system

$$\begin{aligned}\hat{\rho}_A(t) &= Tr_B(\hat{\rho}(t)) \\ &= \sum_b \langle b|\hat{U}(\hat{\rho}(t_0) \bigotimes |0\rangle\langle 0|)\hat{U}^\dagger|b\rangle \\ &= \sum_b \langle b|\hat{U}|0\rangle(\hat{\rho}(t_0)\langle 0|)\hat{U}^\dagger|b\rangle\end{aligned}$$

Let $\hat{M}_b = \langle b|\hat{U}|0\rangle$ , then $\sum_b \hat{M}_b^\dagger \hat{M}_b = I$. So the evolution of system $A$ can be written as

$$\hat{\rho}(t) = \sum \hat{M}_b^\dagger \hat{\rho}(t_0)\hat{M}_b$$

# decoherence

- the evolution operator $M_b$s are not unique, there are many sets satisfy

$$\hat{\$}[\hat{\rho}(t_0)] = \sum \hat{M}_b^\dagger \hat{\rho}(t_0) \hat{M}_b = \sum \hat{N}_a^\dagger \hat{\rho}(t_0) \hat{N}_a$$

the different operators satisfy the condition

$$\hat{M}_b = \sum_a U_{ba} \hat{N}_a$$

where $U$ is a unitary operator

generally, the evolution operator $\hat{\$}$ satisfies the following

  - \$ preserves hermiticity: $\hat{\rho}(t)$ is hermitian if $\hat{\rho}(t_0)$ is.
  - \$ is trace preserving: $Tr(\hat{\rho}(t)) = 1$ if $Tr(\hat{\rho}(t_0)) = 1$
  - \$ is positive: $\hat{\rho}(t)$ is nonnegative if $\hat{\rho}(t_0)$ is
  - \$ is linear
  - \$ is completely positive

# decoherence

- Lindblad equation: from the former operator-sum representation

$$\frac{\partial \hat{\rho}_A(t)}{\partial t} = -\frac{i}{\hbar}[\hat{H}_A, \hat{\rho}_A(t)] + \sum_{\mu \neq 0} (L_\mu \rho L_\mu^\dagger - \frac{1}{2}\rho L_\mu^\dagger L_\mu - \frac{1}{2} L_\mu^\dagger L_\mu \rho) \tag{30}$$

which has a form expression as

$$\frac{\partial \hat{\rho}_A(t)}{\partial t} = L[\hat{\rho}_A(t)] \tag{31}$$

$L$ is named Lindblad operator

- Example: Harmonic oscillator the Hamiltonian of this system is

$$H_A = \hbar \omega a^\dagger a$$

- the damped interaction between the harmonic oscillator and heat bath as

$$H' = \hbar \sum_i g_i (a b_i^\dagger + a^\dagger b_i)$$

If suppose the heat bath is in zero temperature, the system can cascade down by emission photons, but no photon is absorption by the system, then there is only one Lindblad operator

# decoherence

$$L1 = \sqrt{\Gamma} a$$

then the master equation in Lindblad form is

$$\frac{\partial \hat{\rho}_A}{\partial t} = -\frac{i}{\hbar}[\hat{H}_A, \hat{\rho}_A] + \Gamma(a\rho a^\dagger - \frac{1}{2}\rho a^\dagger a - \frac{1}{2}a^\dagger a\rho)$$

transform this equation to the interaction picture as

$$\hat{\rho}(t) = e^{-iH_A t}\hat{\rho}_I(t)e^{iH_A t} \qquad a(t) = e^{-iH_A t}a_I(t)e^{iH_A t}$$

then

$$\dot{\rho_I}(t) = \Gamma(a_I\rho_I a_I^\dagger - \frac{1}{2}\rho_I a_I^\dagger a_I - \frac{1}{2}a_I^\dagger a_I\rho_I)$$

the the evolution of the operator is

$$\begin{aligned}
\frac{\partial\langle a_I\rangle}{\partial t} &= \frac{\partial}{\partial t}Tr(a_I\rho_I) = tr(a_I\dot{\rho}_I) - i\omega Tr(a_I\rho_I) \\
&= \Gamma tr(a_I^2\rho_I a_I^\dagger - \frac{1}{2}a_I\rho_I a_I^\dagger a_I - \frac{1}{2}a_I a_I^\dagger a_I\rho_I) - i\omega Tr(a_I\rho_I) \\
&= \frac{\Gamma}{2}tr([a_I^\dagger, a_I]a_I\rho_I) - i\omega tr(a_I\rho_I) = (-\frac{\Gamma}{2} - i\omega)\langle a_I\rangle
\end{aligned}$$

# decoherence

now get the solution

$$\langle a_I \rangle = e^{-\frac{\Gamma}{2}t} e^{-i\omega t} \langle a_I(0) \rangle \tag{32}$$

it shows that there is an exponentially decay of the average value of $a_I$. On the other hand

$$\begin{aligned}\frac{\partial \langle n \rangle}{\partial t} &= \frac{\partial}{\partial t} tr(a_I \dagger a_I \rho_I) \\ &= \Gamma tr(a_I^\dagger a_I^2 \rho_I a_I^\dagger - \frac{1}{2} a_I \dagger a_I \rho_I a_I^\dagger a_I - \frac{1}{2} (a_I^\dagger a_I)^2 \rho_I) \\ &= \Gamma tr(a^I [a_I^\dagger, a_I] a_I \rho_I) = -\Gamma \langle n \rangle\end{aligned}$$

that is, $\langle n(t) \rangle = e^{-\Gamma t} \langle n(0) \rangle$

- if the interaction with the form

$$H' = \hbar (\sum_i g_i (b_i^\dagger b_i) a^\dagger a$$

the some method as the former case to interaction picture

$$\dot{\rho_I} = \Gamma (a_I^\dagger a_I \rho_I a_I^\dagger a_I - \frac{1}{2} \rho_I (a_I^\dagger a_I)^2 - \frac{1}{2} (a_I^\dagger a_I)^2 \rho_I)$$

# three quantum channels

if expand the density matrix $\rho$ in the occupation number basis as $\rho_{nm}|n\rangle\langle m|$, the master equation will be

$$\begin{aligned} \dot{\rho_{nm}} &= \Gamma(nm - \frac{1}{2}n^2 - \frac{1}{2}m^2)\rho_{nm} \\ &= -\frac{\Gamma}{2}(n-m)^2\rho_{nm} \end{aligned}$$

which give the final results as

$$\rho_{nm}(t) = \rho_{nm}(0)exp[-\frac{1}{2}\Gamma t(n-m)^2] \quad (33)$$

there we will introduce three decoherence models for single qubit, these channels are widely used to discuss the decoherence problems.

- three type of error for channel: bit flip ($\sigma_x$);phase flip ($\sigma_z$);both ($\sigma_y$)
- Depolarizing channel: the qubit in this channel has probability $1 - p$ to remain intact, while with probability $p$ an error occurs. The error can be of any one of three types, and the three types have equally likely.

# three quantum channels

- Unitary representation

$$U : |\psi\rangle \bigotimes |0\rangle \rightarrow \sqrt{1-p}|\psi\rangle \bigotimes |0\rangle + \sum_{i=1}^{3} \sigma_i |\psi\rangle \bigotimes |i\rangle$$

- Kraus representation

$$M_0 = \sqrt{1-p}I, \; M_1 = \sqrt{\frac{p}{3}}, \sigma_x \; M_2 = \sqrt{\frac{p}{3}}\sigma_y, \; M_3 = \sqrt{\frac{p}{3}}\sigma_z$$

then the evolves of the density matrix is

$$\rho' = (1-p)\rho + \frac{p}{3}(\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)$$

- phase-damping channel: This case is particularly instructive because it provides a revealing character of decoherence in realistic physical situation.

# three quantum channels

- Unitary representation

$$|0\rangle_A|0\rangle_E \rightarrow \sqrt{1-p}|0\rangle_A|0\rangle_E + \sqrt{p}|0\rangle_A|1\rangle_E$$
$$|1\rangle_A|0\rangle_E \rightarrow \sqrt{1-p}|1\rangle_A|0\rangle_E + \sqrt{p}|1\rangle_A|2\rangle_E$$

- Kraus operators representation

$$M_0 = \sqrt{1-p}I,\ M_1 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ M_2 = \sqrt{p}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then the evolves of the density matrix is

$$\rho' = \$(\rho) = \sum_i M_i \rho M_i$$
$$= (1-p)\rho + p\begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}$$

# three quantum channels

$$\begin{pmatrix} \rho_{00} & (1-p)\rho_{01} \\ (1-p)\rho_{10} & \rho_{11} \end{pmatrix}$$

the off diagonal terms of this channel will be exponentially suppressed by $(1-p)^n = (1-\Gamma\Delta t)^{t/\Delta t} \to e^{-\Gamma t}$

- Amplitude-damping channel: it is a schematic model of the decay of an ex cited state of a 'two-level' atom due to spontaneous emission of a photon.
- Unitary representation

$$|0\rangle_A|0\rangle_E \to |0\rangle_A|0\rangle_E$$
$$|1\rangle_A|0\rangle_E \to \sqrt{1-p}|1\rangle_A|0\rangle_E + \sqrt{p}|0\rangle_A|1\rangle_E$$

- Kraus operators representation

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

evolution of the density matrix

# three quantum channels

$$\rho' = \$(\rho) = \sum_i M_i \rho M_i$$

$$\begin{pmatrix} \rho_{00} + p\rho_{11} & \sqrt{1-p}\rho_{01} \\ \sqrt{1-p}\rho_{10} & (1-p)\rho_{11} \end{pmatrix}$$

the evolution of this process will make the item $\rho_1 1$ decay as

$$\rho_{11} \rightarrow (1-p)^n \rho_{11}$$

# idea

- the decoherence is the big problem to realize quantum computation, we need some technical tools to overcome this issue.
- In classical computation, if the bit under a binary symmetric channel (it has probability $p$ to flip), then we introduce two more copy bit to encode the information as

$$0 \to 000 \qquad 1 \to 111$$

  this type of decoding is called majority voting, the probability of two or more bits flipped is $3p^2(1-p)+p^3$
- the classical methods can not be extended to quantum case directly
  - Non cloning theorem. there is no way to copy any quantum state exactly.
  - the errors in quantum case are continuous. It seems to determine the errors will need infinitely precision.
  - measurement will destroy quantum information.

# idea

- it looks like the quantum error-correction code is impossible in this way. Fortunately, the quantum error correction code can be designed carefully. As a simple example, we will introduce two simple codes which can detect and correct for some special error

- three qubit bit flip code: the encoding is

$$|0\rangle \to |0_L\rangle \equiv |000\rangle \qquad |1\rangle \to |1_L\rangle \equiv |111\rangle$$

so a single qubit state $a|0\rangle + b|1\rangle$ encode as $a|000\rangle + b|111\rangle$

- error-detecting (syndrome diagnosis): the corresponding measurement operators are

$$\begin{aligned}
P_0 &\equiv |000\rangle\langle 000| + |111\rangle\langle 111| && \text{no error}\\
P_1 &\equiv |100\rangle\langle 100| + |011\rangle\langle 011| && \text{flip on the first qubit}\\
P_2 &\equiv |010\rangle\langle 010| + |101\rangle\langle 101| && \text{flip on the second qubit}\\
P_3 &\equiv |001\rangle\langle 001| + |110\rangle\langle 110| && \text{flip on the third qubit}
\end{aligned}$$

# idea

- another kind of error-detecting (syndrome diagnosis):the corresponding operators are;

$$Z_1Z_2 = (|00\rangle\langle 00| + |11\rangle\langle 11|) \bigotimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|) \bigotimes I$$

$$Z_2Z_3 = I \bigotimes (|00\rangle\langle 00| + |11\rangle\langle 11|) - I \bigotimes (|01\rangle\langle 01| + |10\rangle\langle 10|)$$

  which used to compare the neighboring bit and there are $4$ different outcomes: $(1,1) \rightarrow$ no bit flip; $(-1,1) \rightarrow$ the first bit flip;$(-1,-1) \rightarrow$ the second bit flip;$(1,-1) \rightarrow$ the third bit flip.

- recovery: Using the measurement result, we do the corresponding bit flip. $0$-do nothing;$1$-flip the first qubit;$2$-flip the second qubit;$3$-flip the third qubit

- three qubit phase flip code: the encoding is

$$|0\rangle \rightarrow |0_L\rangle \equiv |+++\rangle \qquad |1\rangle \rightarrow |1_L\rangle \equiv |---\rangle$$

  where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ which can be explained by encoding in bit flip code first, then operate $H$ on each qubit

# idea

- error-detecting (syndrome diagnosis): the corresponding measurement operators are: $X_1X_2, X_2X_3$. There are four different outcomes and the similar correcting processes as before.

- The Shor code: this code can be used to correct any error on a single qubit. The codewords of this encoding is given by

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$
$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

  this encoding is a concatenation process to combine phase flip code and bit flip code.

- error detecting and correcting: this code can used to correct phase and bit flip errors on any qubit. The corresponding operator to measurement and be used as syndromes.

# classical coding theory

The former three examples give some idea about quantum error-correcting code. In the following, we will turn to general theory of constructing quantum codes

- A linear code $C$ encoding $k$ bits information into $n$ bit code space is denoted by an $n \times k$ generator $G$, called $[n, k]$ code.
- the $k$ bit information $x$ (a column vector) is encoded into $n$ bit space by $Gx$
- example: for generator $G = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, the codeword is $G[0] = (0, 0, 0)^T$ and $G[1] = (1, 1, 1)^T$.
- there is another way to define the linear codes by parity check matrix:

$$Hx = 0$$

where $H$ is an $(n - k) \times n$ parity check matrix, $x$ is a $n$-element vector. That is, the codes is the kernel of $H$.

- these two ways can be connected by the following way. $H \to G$: pick $k$ linearly independent vectors $y_1, y_2, ..., y_k$ spanning the kernel of H, the $k$ colums of $G$ is from $y_1$ through $y_k$. $G \to H$:pick $n - k$ linearly independent vectors$y_1, y_2, ..., y_{n-k}$ orthogonal to the columns of $G$, $H$'s rows are $y_1^T, y_2^T, ..., y_{n-k}^T$.

# classical coding theory

- example of $G \to H$: for generator $G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ the parity check matrix is construct by two linearly independent vectors orthogonal to the column of $G$,

$$H \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

now we introduce several useful concepts in linear code theory

- error syndrome: let the code of $x$ is $y = Gx$, if there is an error $e$ make $y' = y + e$. The error syndrome is $Hy' = He$, different error $e_j$ have different syndrome $He_j$.
- Hamming distance between two codes $x$ and $y$ is defined to be the number of bits at which $x$ and $y$ differ, denoted as $d(x, y)$
- Hamming weight of a code $x$ is defined to be the distance from the string zero, that is the number of $1$ in $x$.
- distance of a code is defined to be

$$d(C) \equiv \min_{x,y \in C, x \neq y} d(x, y) = \min_{x \in C, x \neq 0} wt(x)$$

# CSS codes

- if the code satisfy $d(C) \geq 2t+1$, it is able to correct errors on up to $t$ bits. That is, the corrupted encoded message $y'$ is the unique codeword satisfying $d(y, y') \leq t$.

CSS codes is an important subclass of general quantum codes which can be constructed through classical linear codes

- $C_1$ and $C_2$ are $[n, k_1]$ and $[n, k_2]$ linear codes such that $C_2 \in C_1$ and $C_1$ and $C_2^{\perp}$ both correct $t$ errors. Then we can define an $[n, k_1 - k_2]$ quantum code which can correct errors on $t$ qubit. Suppose $x \in C_1$ is a codeword in $C_1$, the quantum state $|x + C_2\rangle$ can be defined as:

$$|x + C_2\rangle \equiv \frac{1}{|C_2|} \sum_{y \in C_2} |x + y\rangle$$

- It is clear from the definition that $|x + C_2\rangle = |x' + C_2\rangle$ when $x' - x \in C_2$. That is, the code is dependent of the coset of $C_1/C_2$
- For two different coset of $C_2$, the corresponding states $|x + C_2\rangle$ and $|x' + C_2\rangle$ are orthogonal.
- the dimension of CSS code is$|C_1|/|C_2| = 2^{k_1 - k_2}$, so the CSS code is an $[n, k_1 - k_2]$ quantum code.

# CSS codes

The CSS code can error-correct up to $t$ bit and phase flip error. Suppose the bit flip errors denoted by $e_1$ with 1s where the bit flip occurred,the phase flip errors denoted by $e_2$ with 1s where the phase flip occurred. The corrupted state of the initial state $|x + C_2\rangle$ is

$$\frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} (-1)^{(x+y).e_2} |x + y + e_1\rangle$$

To detect and correct bit flip errors:

- Introduce an ancilla (initially in zero) containing enough sufficient qubit to store the syndrome for code $C_1$
- Using reversible computation to apply parity matrix $H_1$ for the code $C_1$ to make

$$|x + y + e_1\rangle|0\rangle \to |x + y + e_1\rangle|H_1(x + y + e_1)\rangle = |x + y + e_1\rangle|H_1(e_1)\rangle$$

and get the state

$$\frac{1}{\sqrt{|C_2|}} \sum_{y \in C_2} (-1)^{(x+y).e_2} |x + y + e_1\rangle|H_1(e_1)\rangle$$

# CSS codes

- measurement the ancilla to get the result $He_1$ and discard the ancilla, and get the state

$$\frac{1}{\sqrt{|C_2|}}\sum_{y\in C_2}(-1)^{(x+y).e_2}|x+y+e_1\rangle$$

- knowing the error syndrome $He_1$ infer the error $e_1$ since $C_1$ can correct up to $t$ errors.
- recovery: apply NOT gates to the qubits at which positions in the error $e_1$. Then get the state

$$\frac{1}{\sqrt{|C_2|}}\sum_{y\in C_2}(-1)^{(x+y).e_2}|x+y\rangle$$

To detect and correct phase flip error

- to apply Hadamard gates to each qubit, the state will be

$$\frac{1}{\sqrt{|C_2|2^n}}\sum_{z}\sum_{y\in C_2}(-1)^{(x+y).(e_2+z)}|z\rangle$$

$$= 1/\sqrt{|C_2|2^n}\sum\sum(-1)^{(x+y).z'}|z'+e_2\rangle \quad \text{setting } z'\equiv z+e_2$$

# CSS codes

- supposing $z' \in C_2^{\perp}$, the state can be written as

$$\frac{1}{\sqrt{|C_2|2^n}} \sum_{z' \in C_2^{\perp}} (-1)^{x.z'} |z' + e_2\rangle$$

this is the form of the bit flip error of $e_2$ which can be correct by the same way as the former way. And get the state

$$\frac{1}{\sqrt{|C_2|2^n}} \sum_{z' \in C_2^{\perp}} (-1)^{x.z'} |z'\rangle$$

- applying Hadamard gates to each qubit and complete the error-correction.

A CSS code example:Steane code
a $[7, 4, 3]$ Hamming code labeled by $C$ with parity check matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

# CSS codes

- Let $C_1 \equiv C$ and $C_2 \equiv C^{\perp}$. We can verify that the parity check matrix of $C_2$ is:

$$H[C_2] = G[C_1]^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

- it can be verified that the span of the rows of $H[C_2]$ strictly contains the span of the rows of $H[C_1]$. And the $C_1$ and $C_2$ are distance $3$ codes which can correct errors on $1$ bit.
- Based on the method of CSS code constructing, $C_1$ and $C_2$ can construct a $[7, 1]$ quantum code named *Steane code*:

$$\begin{aligned} |0 + C_2\rangle \to |0_L\rangle \equiv & \frac{1}{\sqrt{8}}[|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\ & + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1100110\rangle] \\ |0 + C_2\rangle \to |1_L\rangle \equiv & \frac{1}{\sqrt{8}}[|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle \\ & + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle] \end{aligned}$$

# stabilizer code

- Suppose $S$ is a subgroup of pauli group $G_n$ and define $V_s$ to be the set of qubit states which are fixed by every element of $S$.
- Suppose $S = \langle g_1, ..., g_l \rangle$ and $g_1, ..., g_l$ to be independent. we can present the generators of $S$ in parity check form which is a $l \times 2n$ matrix $M$. The left $l \times n$ matrix corresponding the operator of $x$ and the right $l \times n$ matrix corresponding the operator of $z$.
- Let $S = \langle g_1, ..., g_l \rangle$ be such that $-I \bar{\in} S$, the generators are independent if and only if the rows of the corresponding check matrix are linearly independent.
- example of stabilizer code: the stabilizer for the Steane qubit code

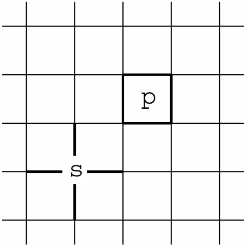
$$
\begin{array}{llll}
\mathrm{g}_1: & I\ I\ I\ X\ X\ X\ X & g_2: & I\ X\ X\ I\ I\ X\ X \\
\mathrm{g}_3: & X\ I\ X\ I\ X\ I\ X & g_4: & I\ I\ I\ Z\ Z\ Z\ Z \\
\mathrm{g}_5: & I\ Z\ Z\ I\ I\ Z\ Z & g_6: & Z\ I\ Z\ I\ Z\ I\ Z
\end{array}
$$

# stabilizer code

the corresponding check matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \tag{34}$$

# toric code




we can define a set of stabilizers as

$$A_s = \prod_{j \in star(s)} \sigma_j^x \qquad B_p = \prod_{j \in boundary(p)} \sigma_j^z$$

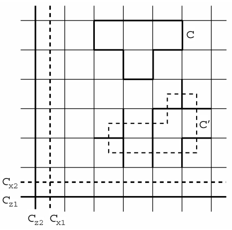
define the vector space $|\psi\rangle$ as

$$A_s|\psi\rangle = |\psi\rangle \qquad B_s|\psi\rangle = |\psi\rangle \tag{35}$$

when the corresponding geometry have genus, the corresponding vector space is topological protected.

# toric code

- the operating of the stabilizer code




- the topological of the code