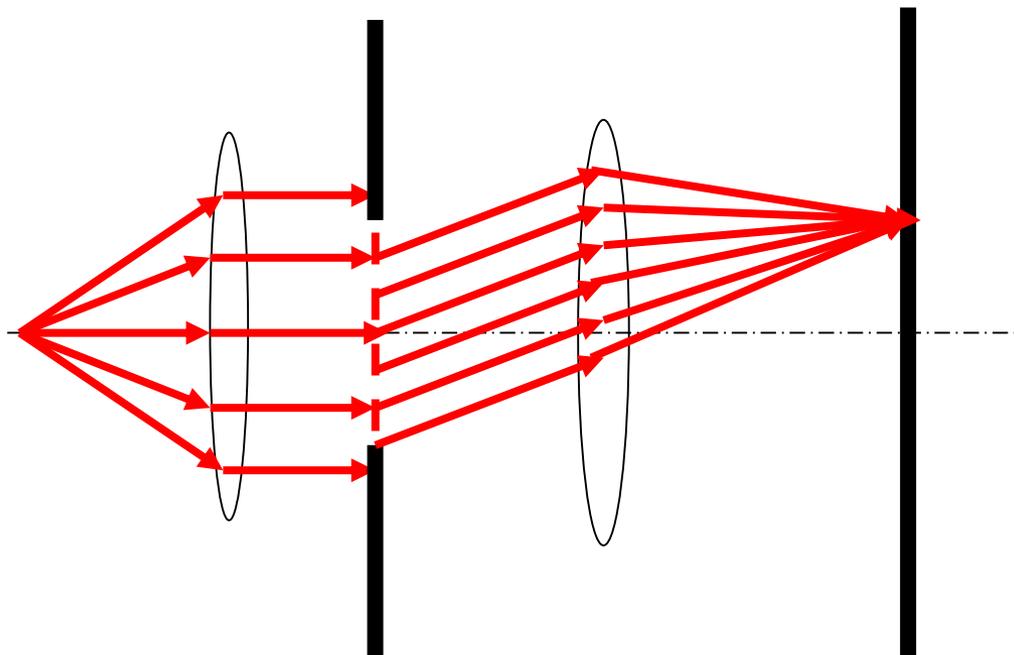


# 4-03夫琅和费(Fraunhofer) 单缝和矩孔衍射

- 1 实验装置
- 2 单缝衍射的强度公式
- 3 单缝衍射因子的特点
- 4 夫琅和费矩孔衍射
- 5 衍射反比关系的意义

# 1. 实验装置

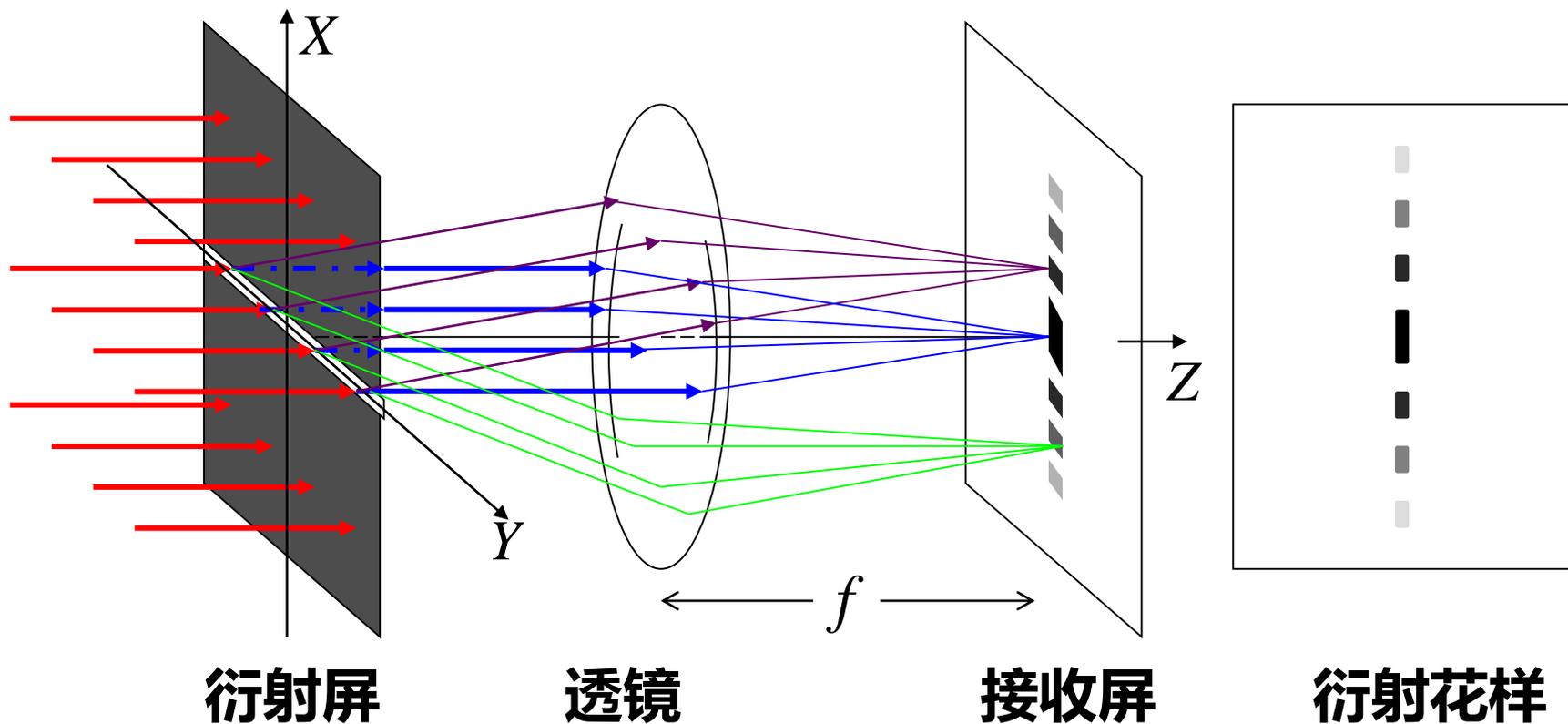
- 衍射装置



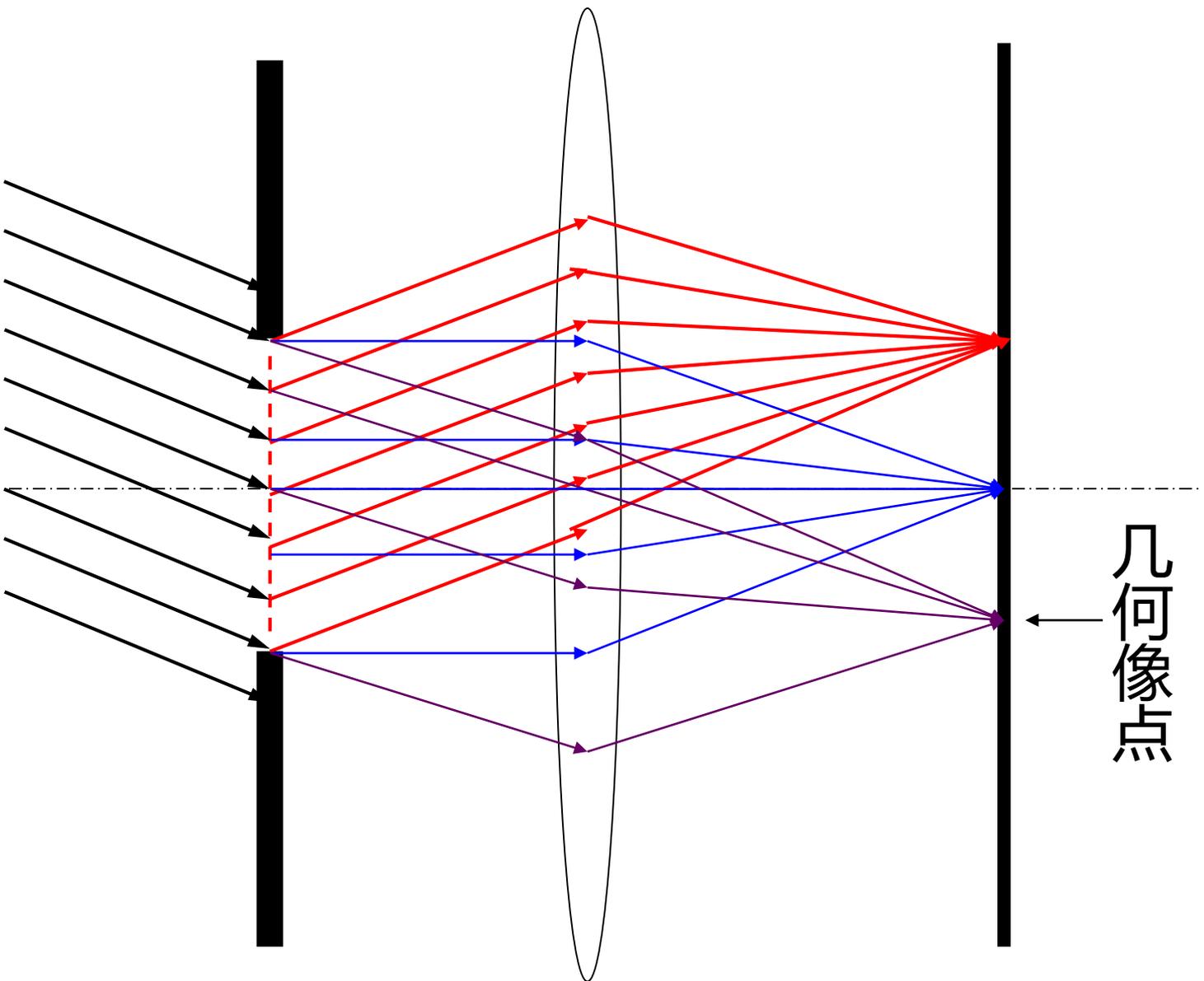
- 平行光入射，用凸透镜成像于像方焦平面。
- 相当于各点发出的次波汇聚于无穷远处。即是平行光的相干叠加。

# 衍射花样

- 在**焦平面**上汇聚（相遇）的光，是从狭缝发出的**相互平行**的**次波**



入射光不一定平行于光轴



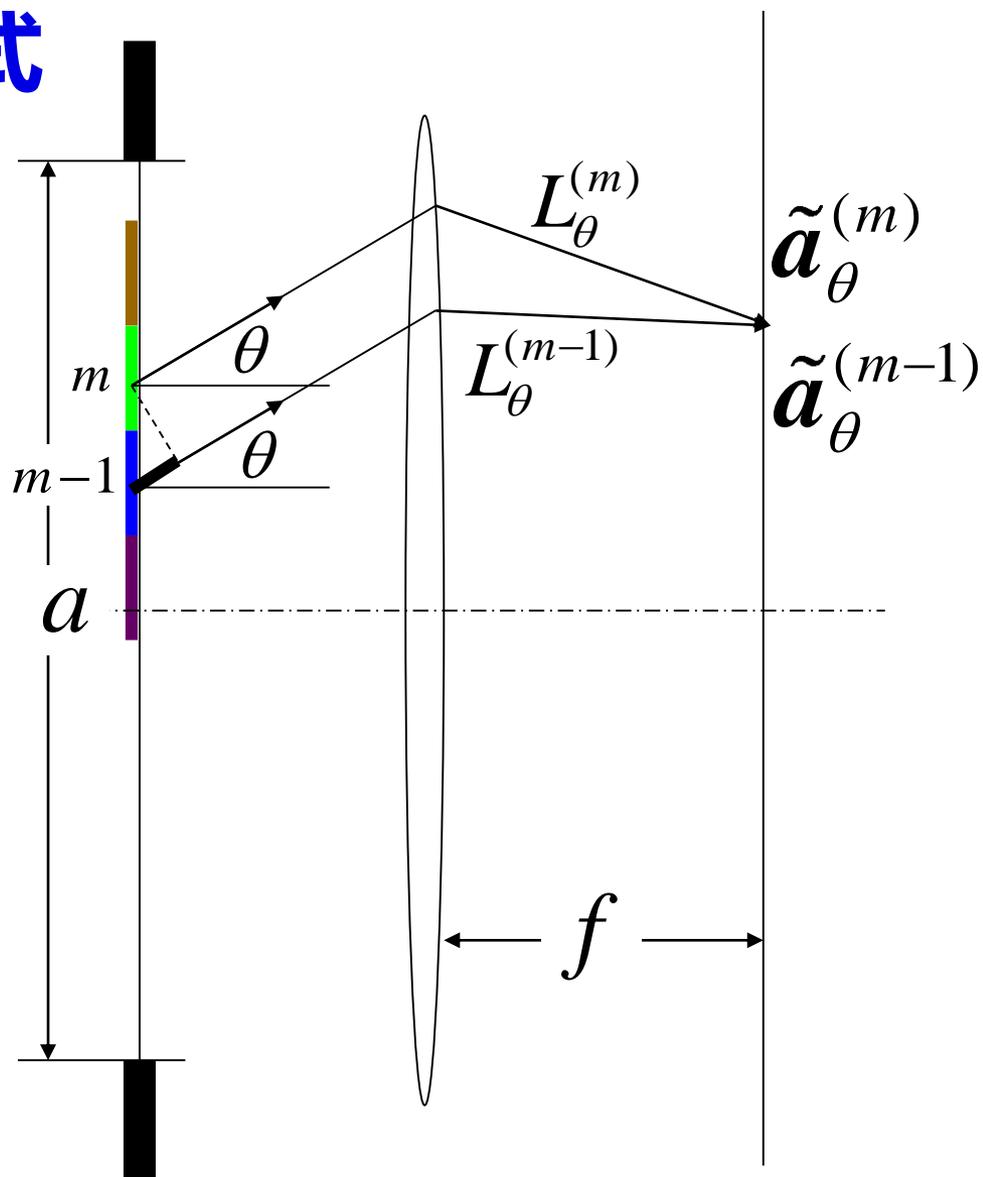
几何像点

## 2. 单缝衍射的强度公式

- 求解积分公式

### 一、振幅矢量法

- 将波前 $N$ 等分
- 每个面元宽度为 $a/N$
- $\tilde{\mathbf{a}}_{\theta}^{(m)}$ : 第 $m$ 个面元发出的次波的复振幅
- $L_{\theta}^{(m)}$ : 第 $m$ 个面元发出的次波的光程



相邻两单元次波的光程差

$$\Delta L = \frac{a \sin \theta}{N}$$

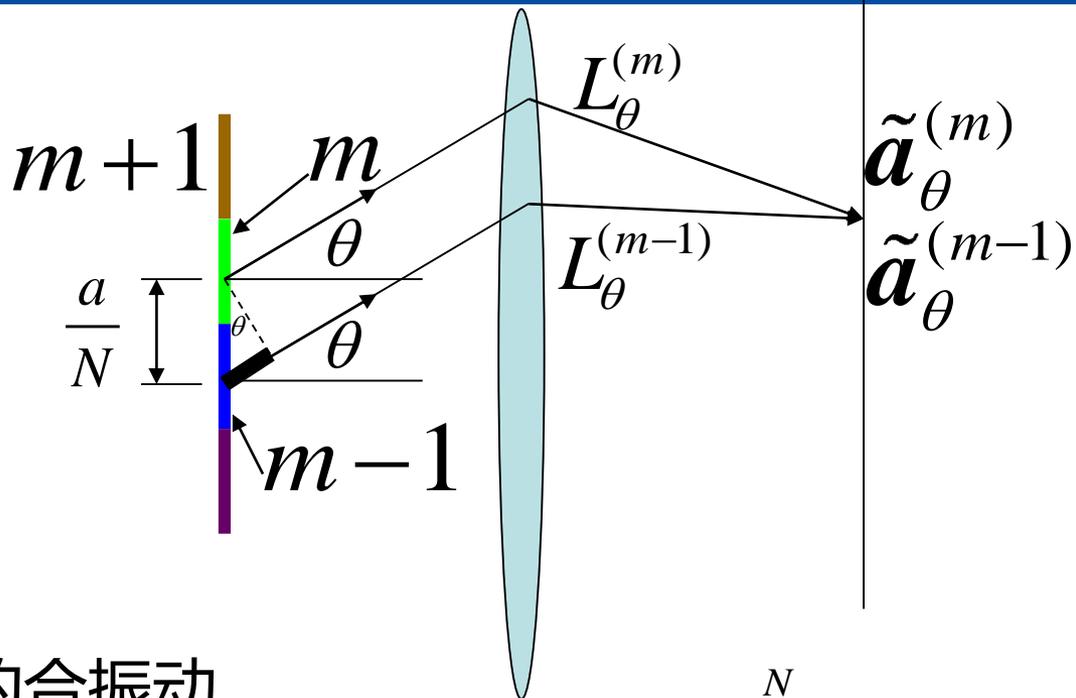
相邻两单元次波的相位差

$$\Delta \varphi = k \Delta L = \frac{ka \sin \theta}{N}$$

沿 $\theta$ 方向的次波在接收屏上的合振动

在傍轴条件下，忽略倾斜因子的影响  
各个单元沿不同方向的次波振幅相等

$$\tilde{U}(P) = K \iint_{\Sigma} \tilde{U}(Q) F(\theta_0, \theta) \frac{e^{ikr}}{r} d\Sigma$$



$$\tilde{A}_{\theta} = \sum_{m=1}^N \tilde{a}_{\theta}^{(m)}$$

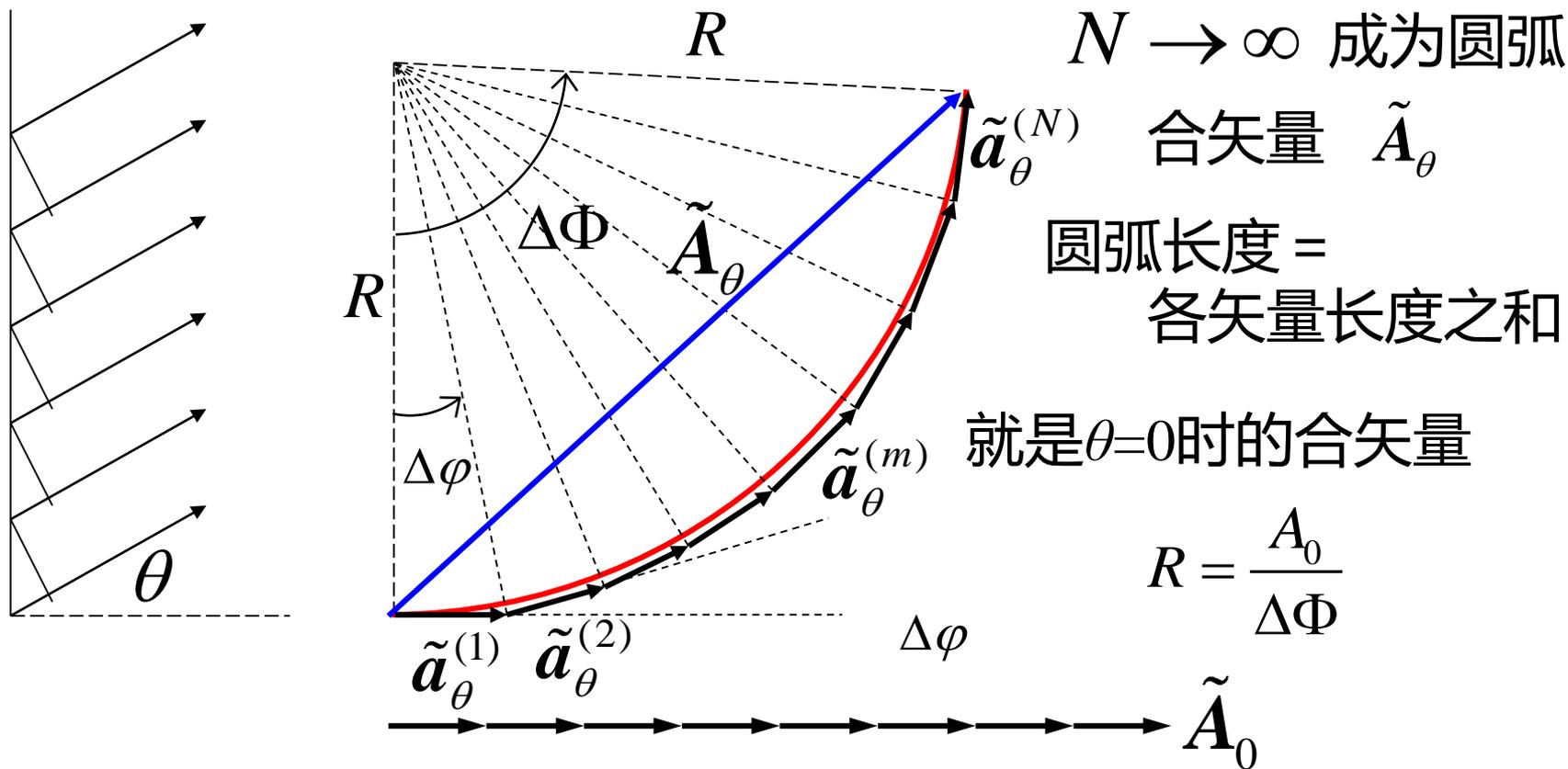
$$\frac{F(\theta_0, \theta)}{r} = \frac{1}{L_0}$$

各个面元的瞳函数相等

# 振幅矢量求和

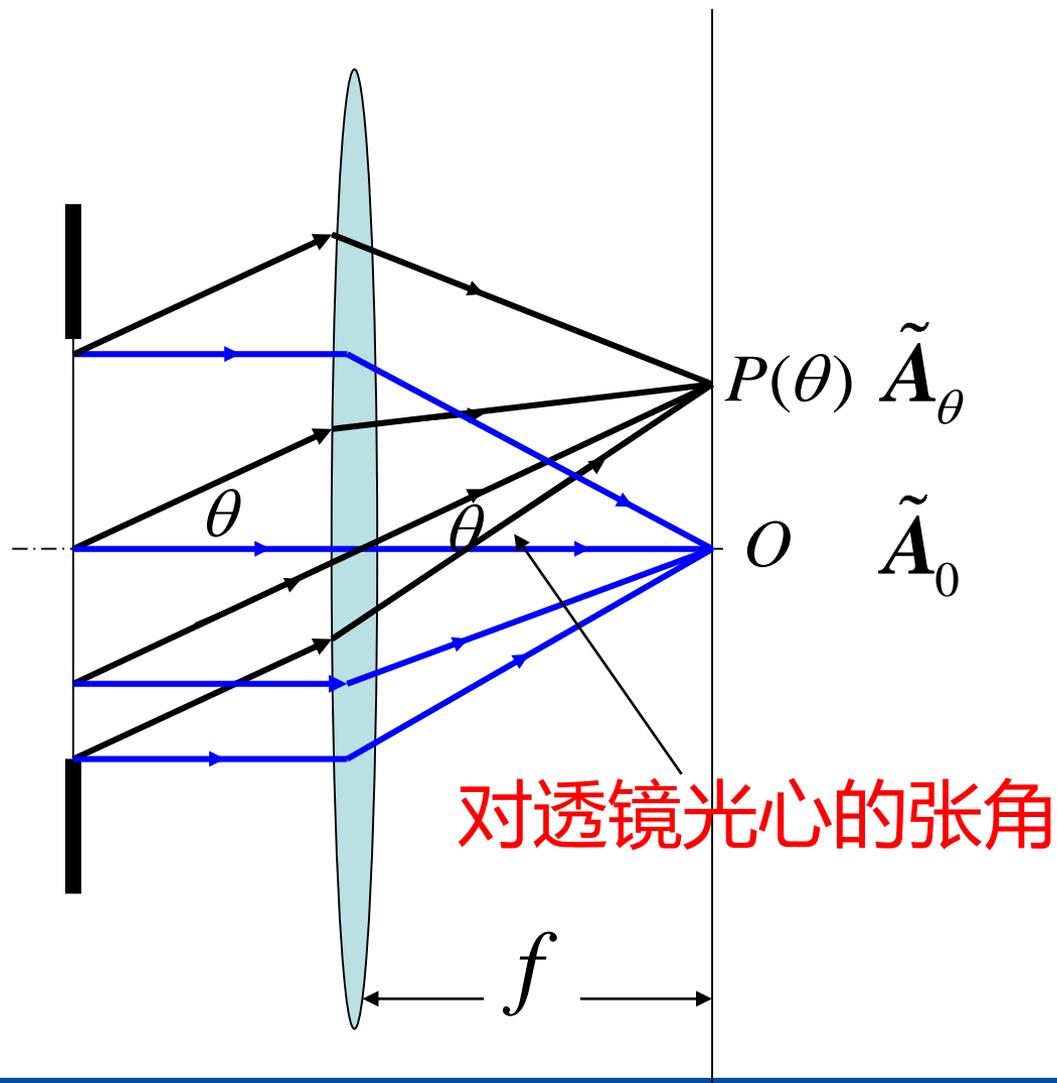
$N$ 个矢量，每个依次转过  $\Delta\varphi$

共转过  $\Delta\Phi = N\Delta\varphi$  构成一段圆弧的 $N$ 条弦





# 各个参数的物理意义



$$\tilde{A}_\theta = \tilde{A}_0 \frac{\sin u}{u}$$

$$I_\theta = |\tilde{A}_\theta|^2 = A_0^2 \frac{\sin^2 u}{u^2}$$
$$= I_0 \frac{\sin^2 u}{u^2}$$

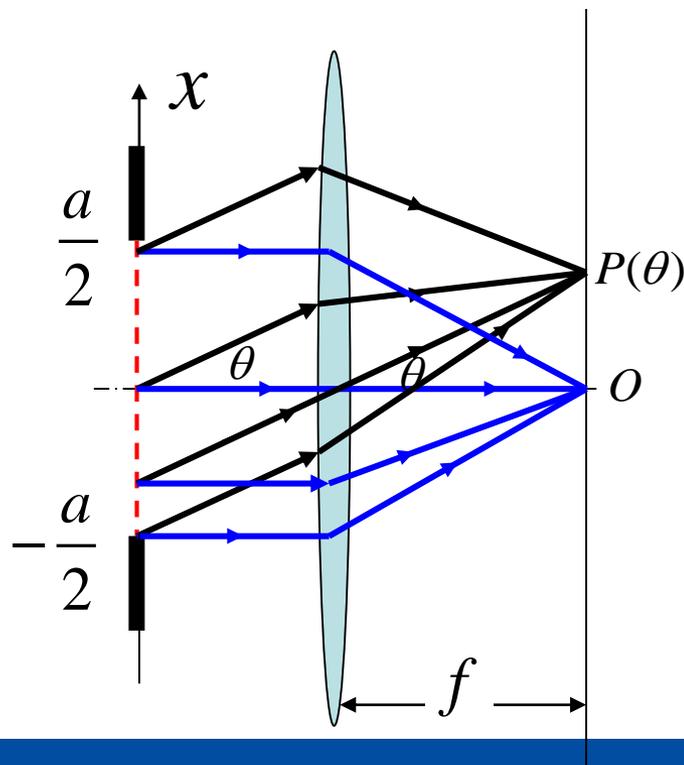
$I_0$  点的光强

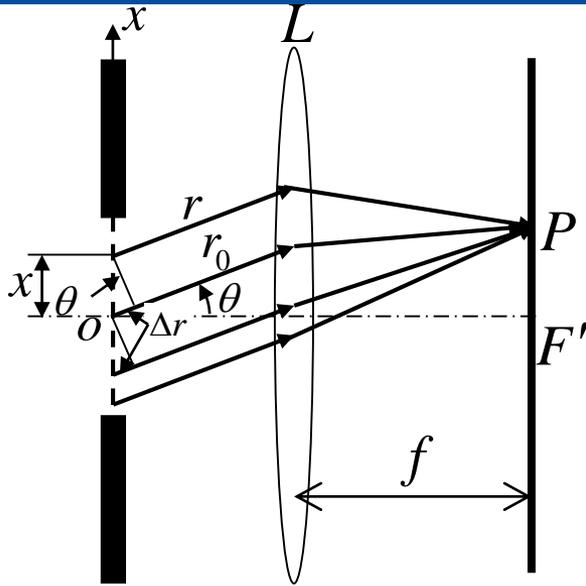
## 二、积分方法

$$\tilde{U}(P) = K \iint_{\Sigma} \tilde{U}(Q) F(\theta_0, \theta) \frac{e^{ikr}}{r} d\Sigma$$

- $P(\theta)$ 点的次波来自同一方向，倾斜因子相同。
- 不同方向的光，满足近轴条件，倾斜因子为常数1。
- 瞳函数为常数
- 积分简化

$$\begin{aligned} \tilde{U}(P) &= K \frac{\tilde{U}(Q) F(\theta_0, \theta)}{L_0} \iint_{\Sigma} e^{ikr} d\Sigma \\ &= \frac{K \tilde{U}(Q)}{L_0} \int_{-a/2}^{a/2} e^{ikr} dx \end{aligned}$$





$$\tilde{U}(P) = K \iint_{\Sigma} \tilde{U}_0(Q) \frac{e^{ikr}}{r} d\Sigma = \frac{\tilde{K}_0}{L_0} \int_{-a/2}^{a/2} e^{ikr} dx$$

$$F(\theta_0, \theta) = 1 \quad \tilde{K}_0 = K\tilde{U}_0(Q) \quad \boxed{r = r(x) = ?}$$

$$\Delta r = -x \sin \theta \quad r = r_0 + \Delta r = r_0 - x \sin \theta$$

$$\tilde{U}(P) = \frac{\tilde{K}_0}{L_0} e^{ikr_0} \int_{-a/2}^{a/2} e^{-ikx \sin \theta} dx = \frac{\tilde{K}_0 e^{ikr_0}}{L_0} \frac{1}{-ik \sin \theta} \left[ e^{-ik \frac{a}{2} \sin \theta} - e^{ik \frac{a}{2} \sin \theta} \right]$$

$$= \frac{\tilde{K}_0 e^{ikr_0}}{L_0} \frac{-2i \sin\left(\frac{ka}{2} \sin \theta\right)}{-ik \sin \theta} = aK\tilde{U}_0(Q) \frac{e^{ikr_0}}{L_0} \frac{\sin\left(\frac{1}{2} ka \sin \theta\right)}{\frac{1}{2} ka \sin \theta} = \tilde{U}_0 \frac{\sin u}{u}$$

$$\left( \tilde{U}_0 = aK\tilde{U}_0(Q) \frac{e^{ikr_0}}{L_0}, \quad u = \frac{\pi a \sin \theta}{\lambda} \right)$$

$$\tilde{U}(P) = aK\tilde{U}_0(Q) \frac{e^{ikr_0}}{L_0} \frac{\sin(\frac{1}{2}ka \sin \theta)}{\frac{1}{2}ka \sin \theta} = \tilde{U}_0 \frac{\sin u}{u}$$

$$K\tilde{U}_0(Q) \frac{e^{ikr_0}}{L_0}$$

狭缝上Q点发出的次波在几何像点所引起的复振幅

$$\tilde{U}_0 = aK\tilde{U}_0(Q) \frac{e^{ikr_0}}{L_0}$$

通过整个狭缝的次波在几何像点上复振幅

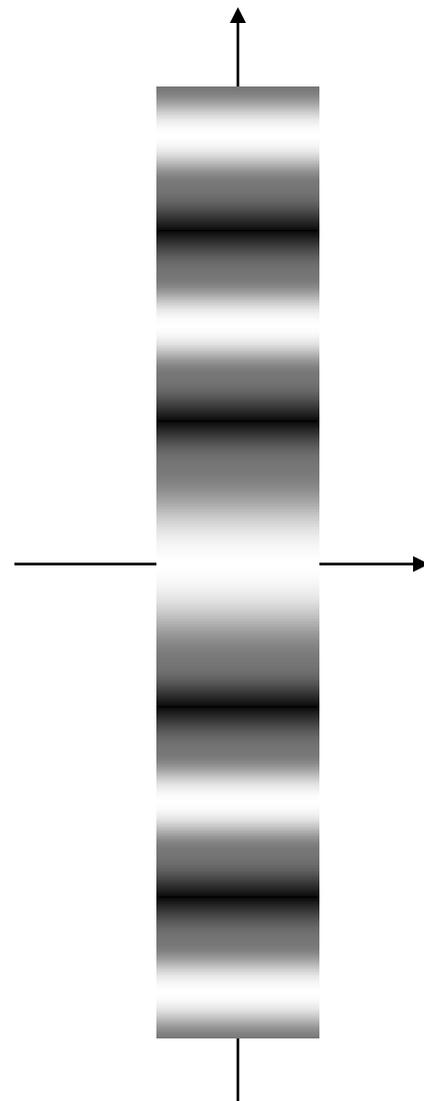
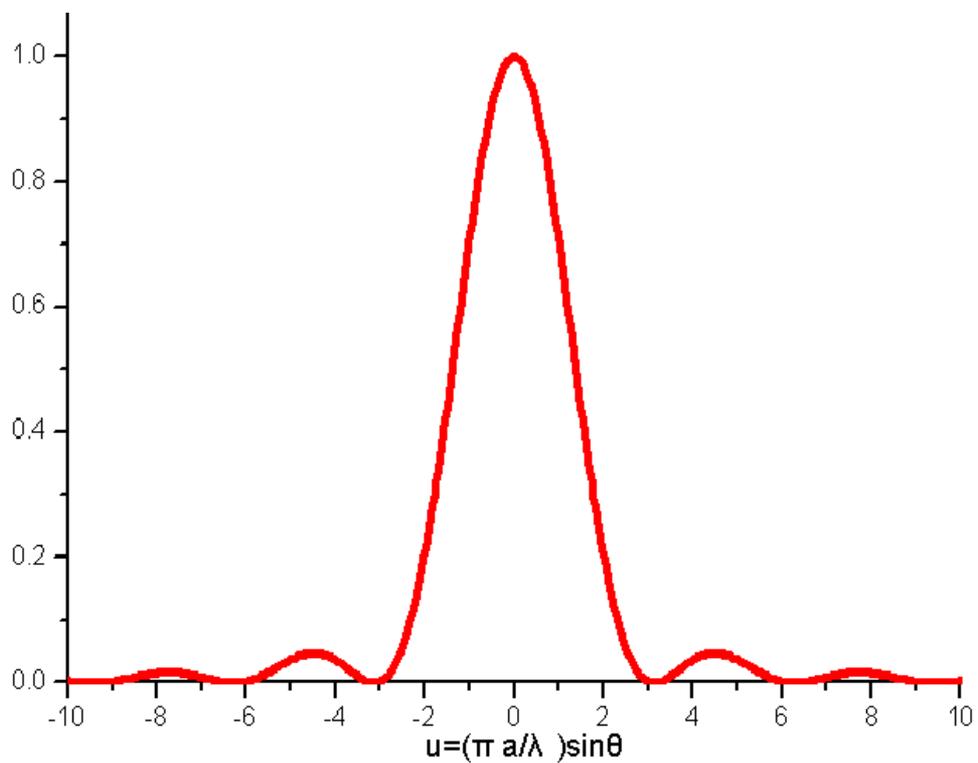
$$u = \frac{1}{2}ka \sin \theta = \frac{\pi a}{\lambda} \sin \theta$$

强度分布  $I(P) = I_0 \frac{\sin^2 u}{u^2}$   $\frac{\sin^2 u}{u^2}$  称作单缝 (单元) 衍射因子

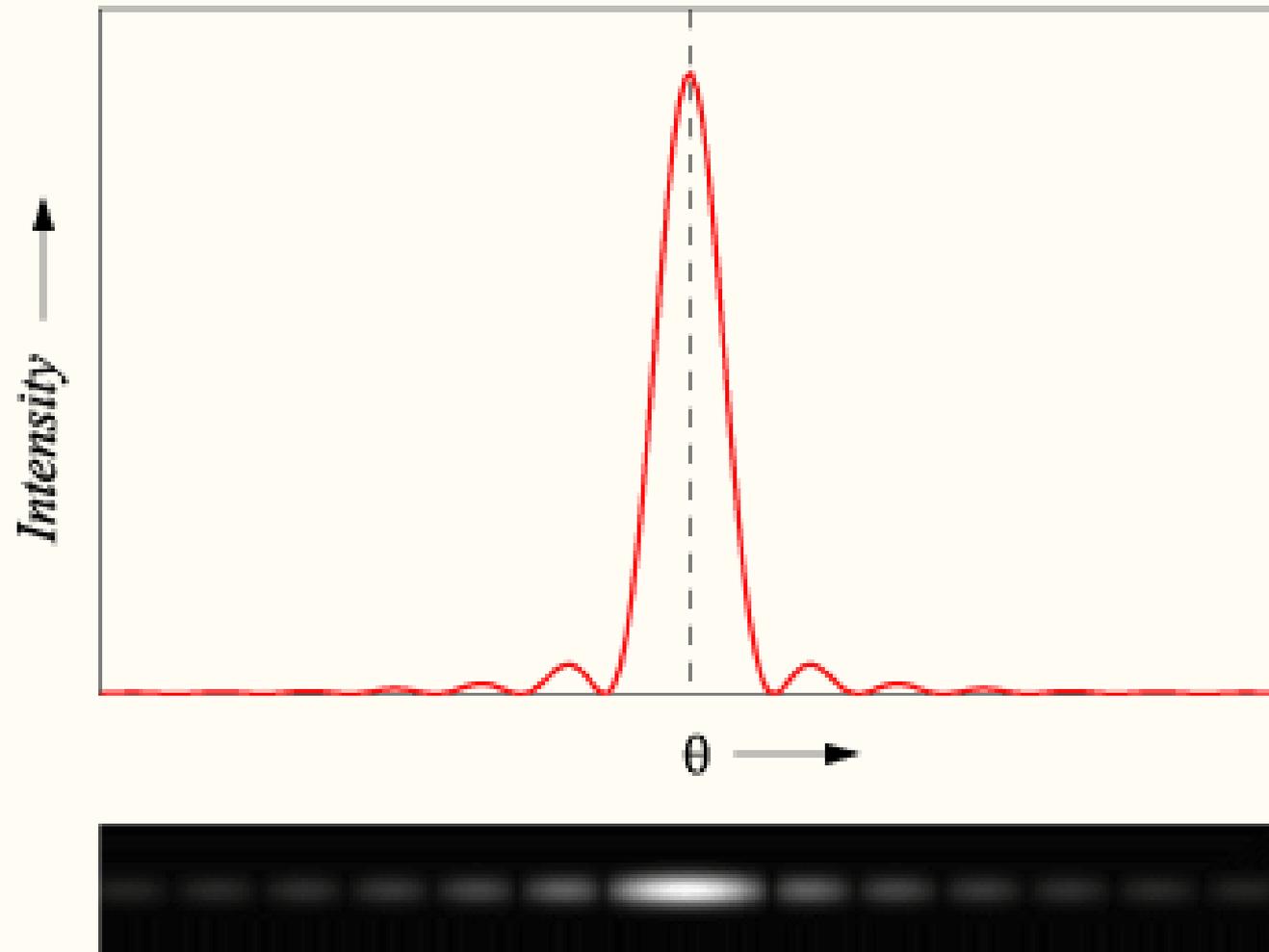
$$I_0 = \tilde{U}_0 \tilde{U}_0^*$$

几何像点处的光强

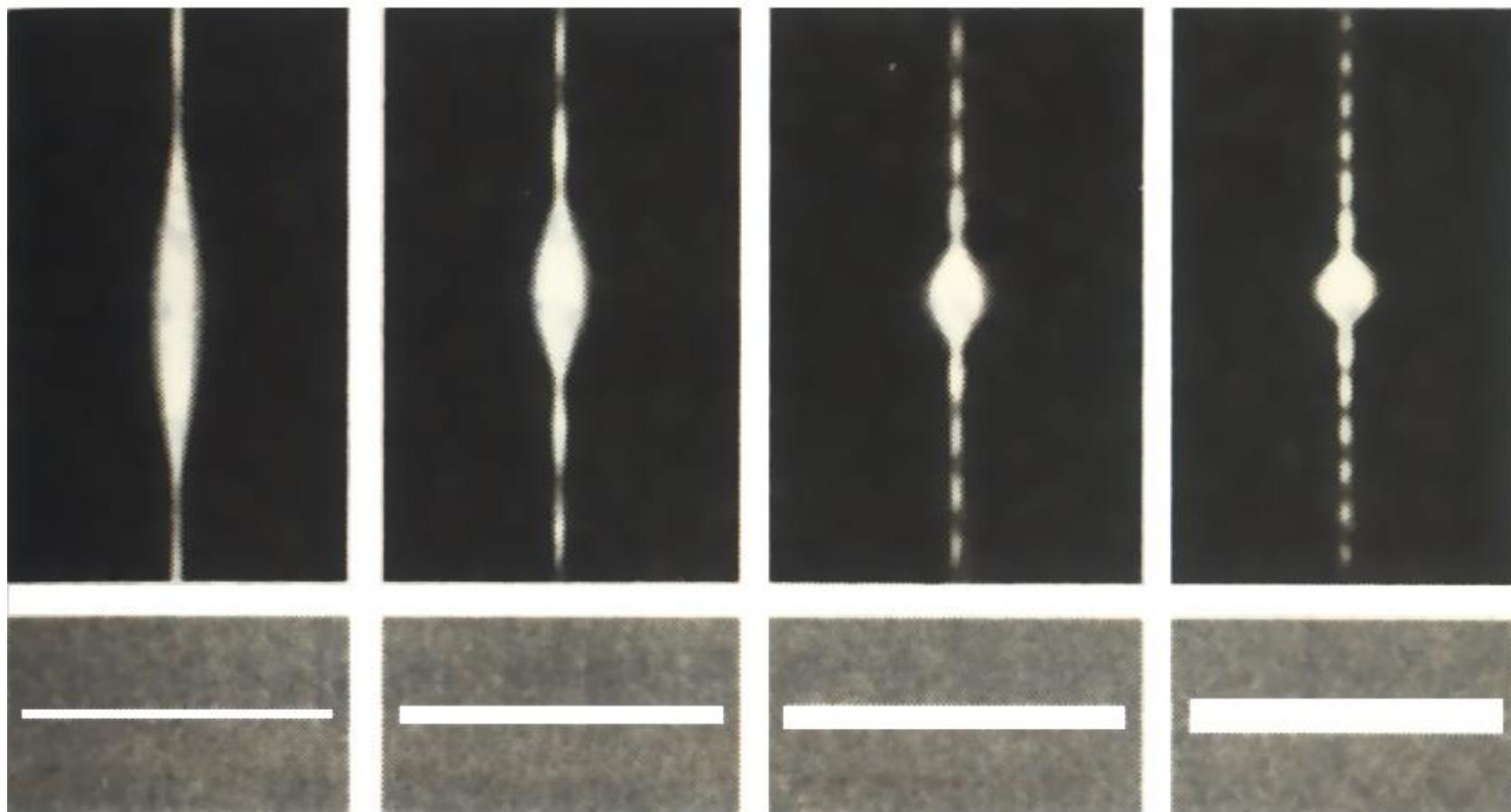
# 夫琅和费单缝衍射花样



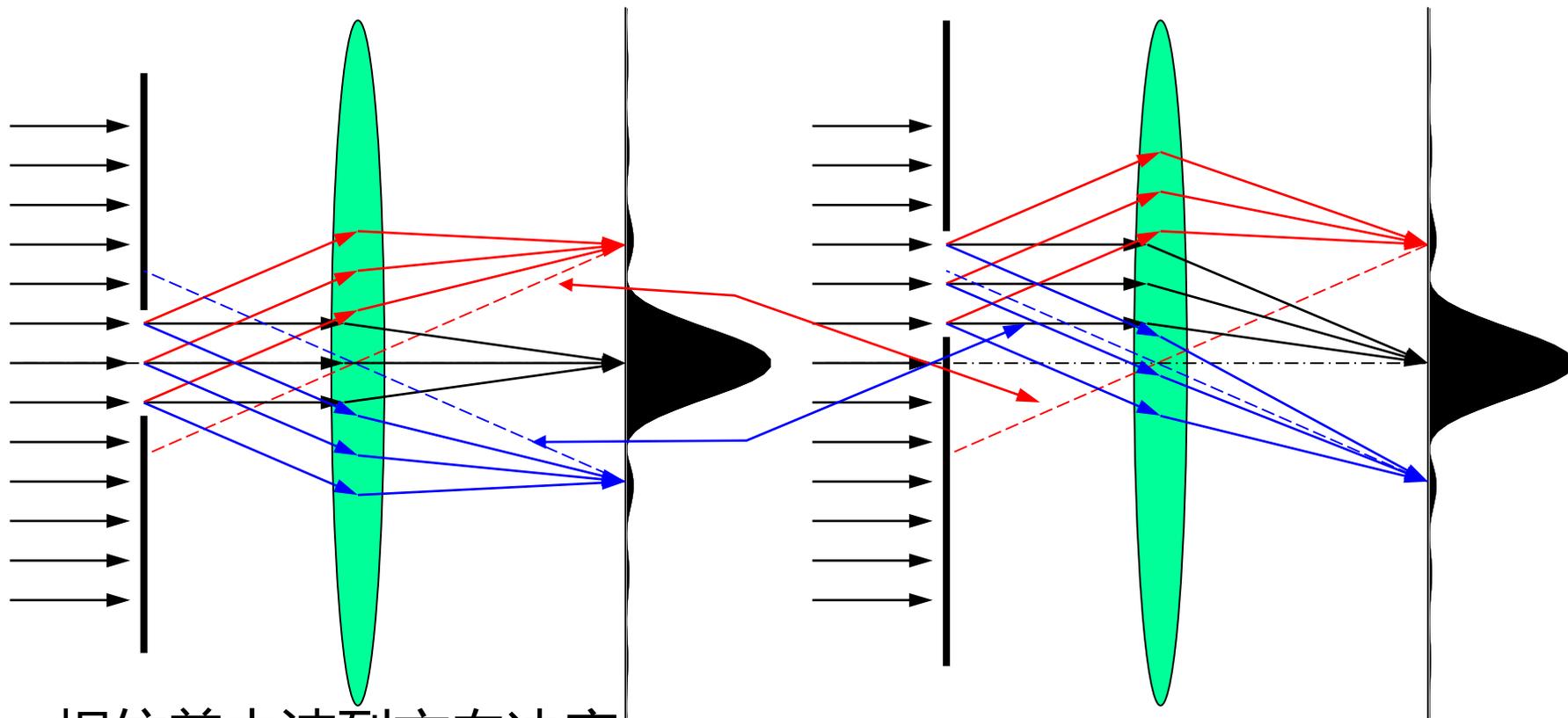
## *Single-slit diffraction pattern*



# 不同宽度狭缝的衍射花样



# 狭缝移动的影响

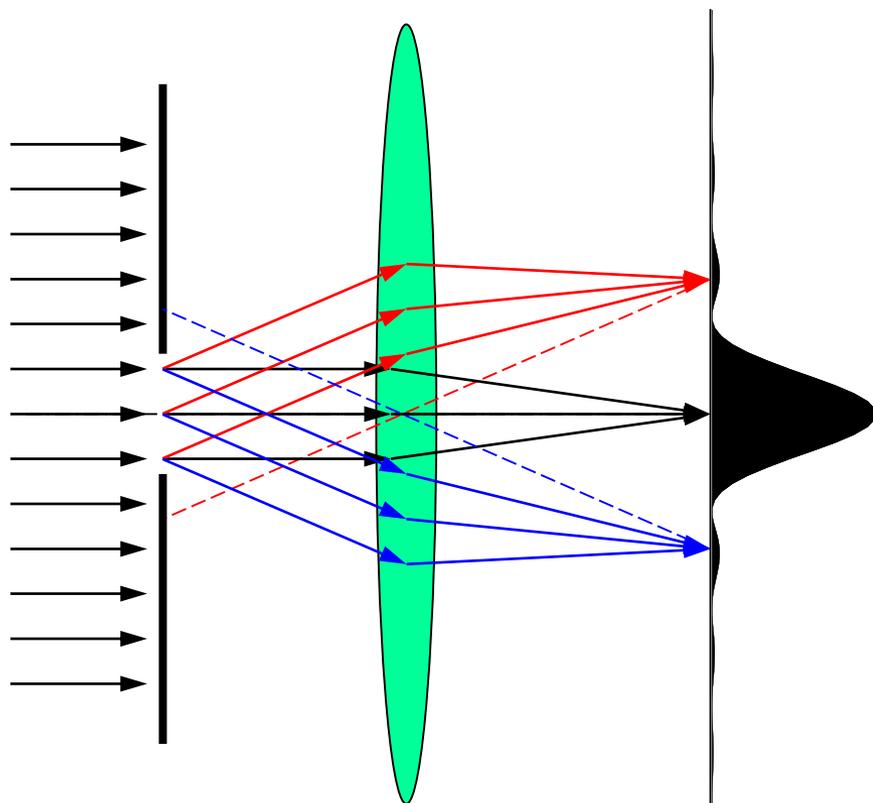


相位差由波列方向决定

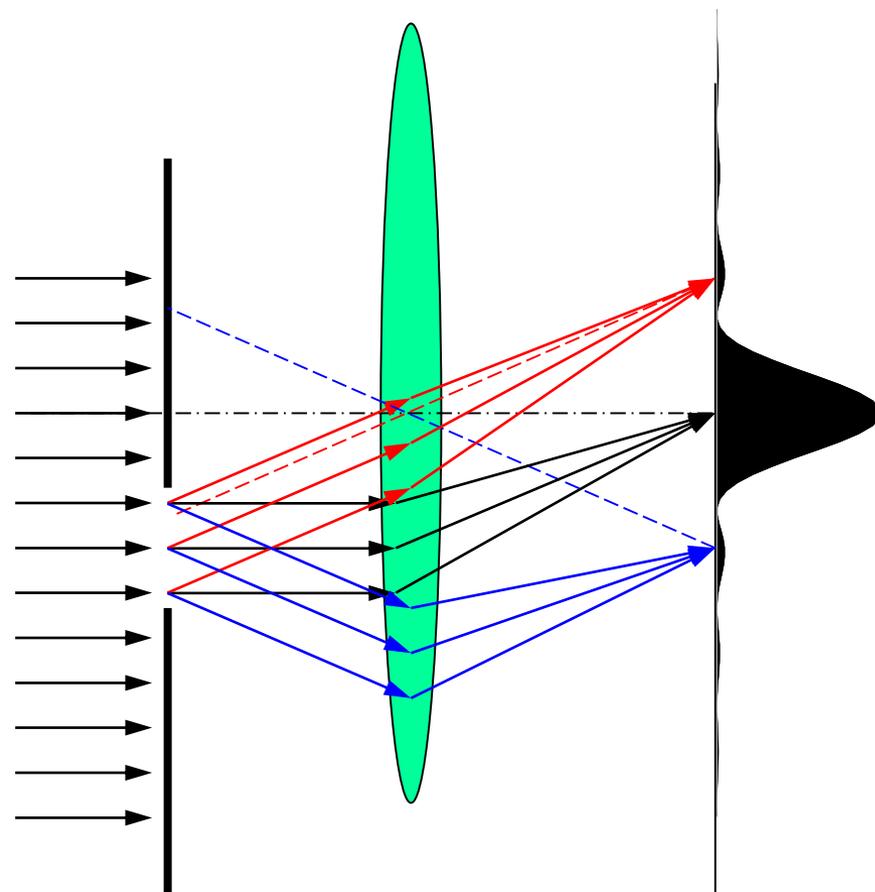
焦平面上衍射强度也由波列方向决定

狭缝上下移动，衍射花样不变

# 透镜移动的影响



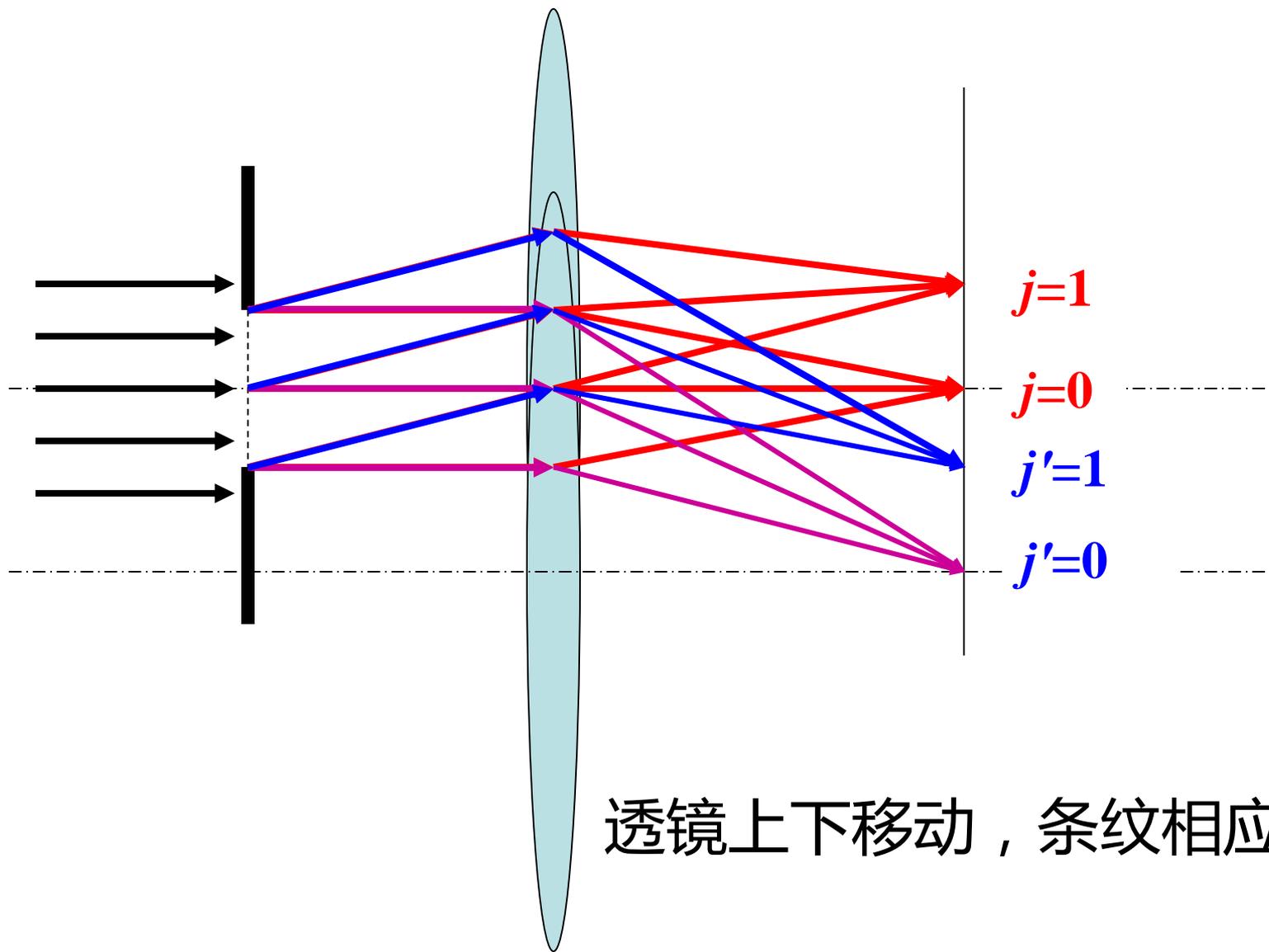
相位差由波列方向决定



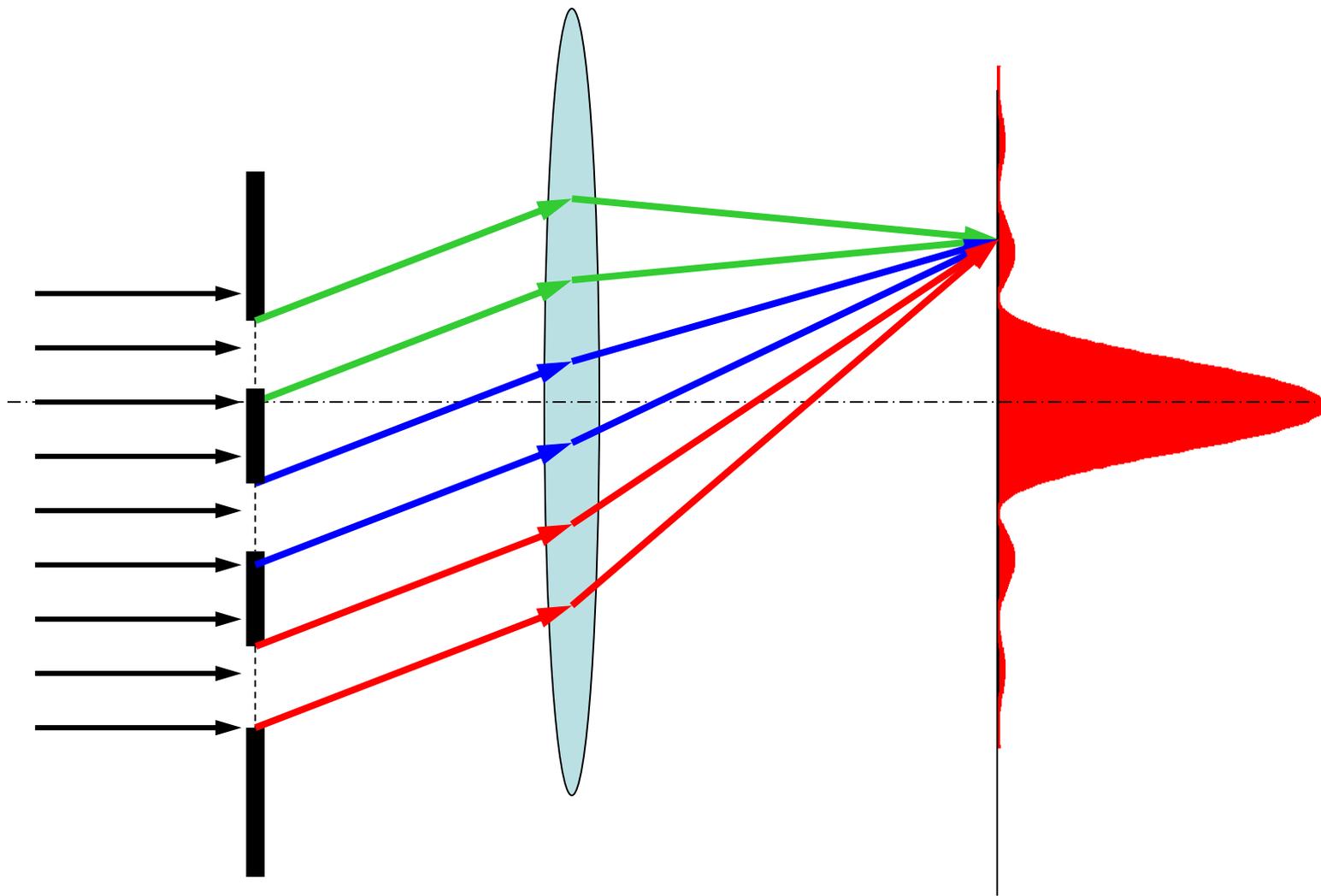
透镜光轴是系统的对称轴

焦平面上衍射强度也由波列方向决定

衍射花样随光轴移动

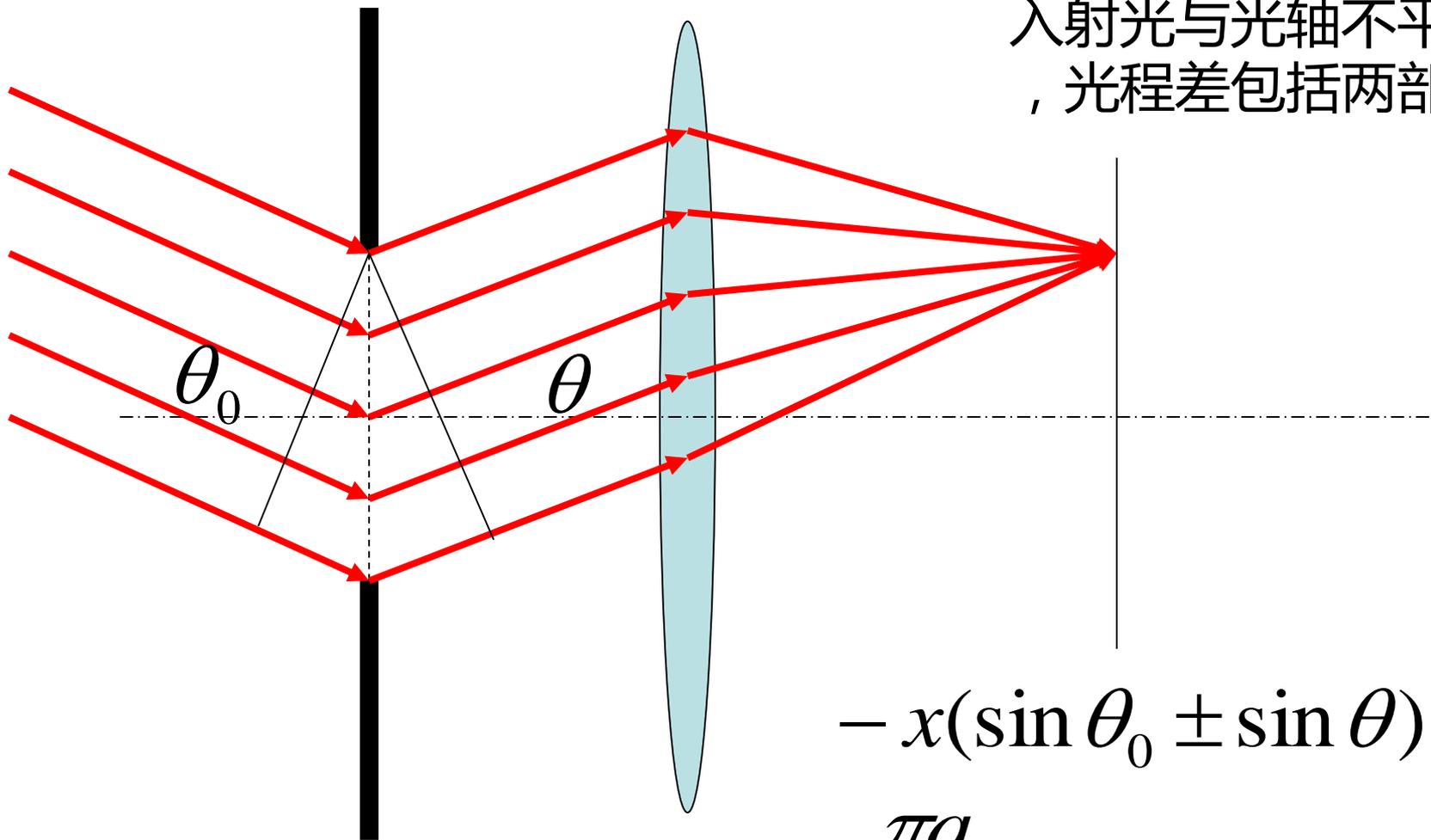


透镜上下移动，条纹相应移动



相互平行的狭缝，衍射条纹完全重合

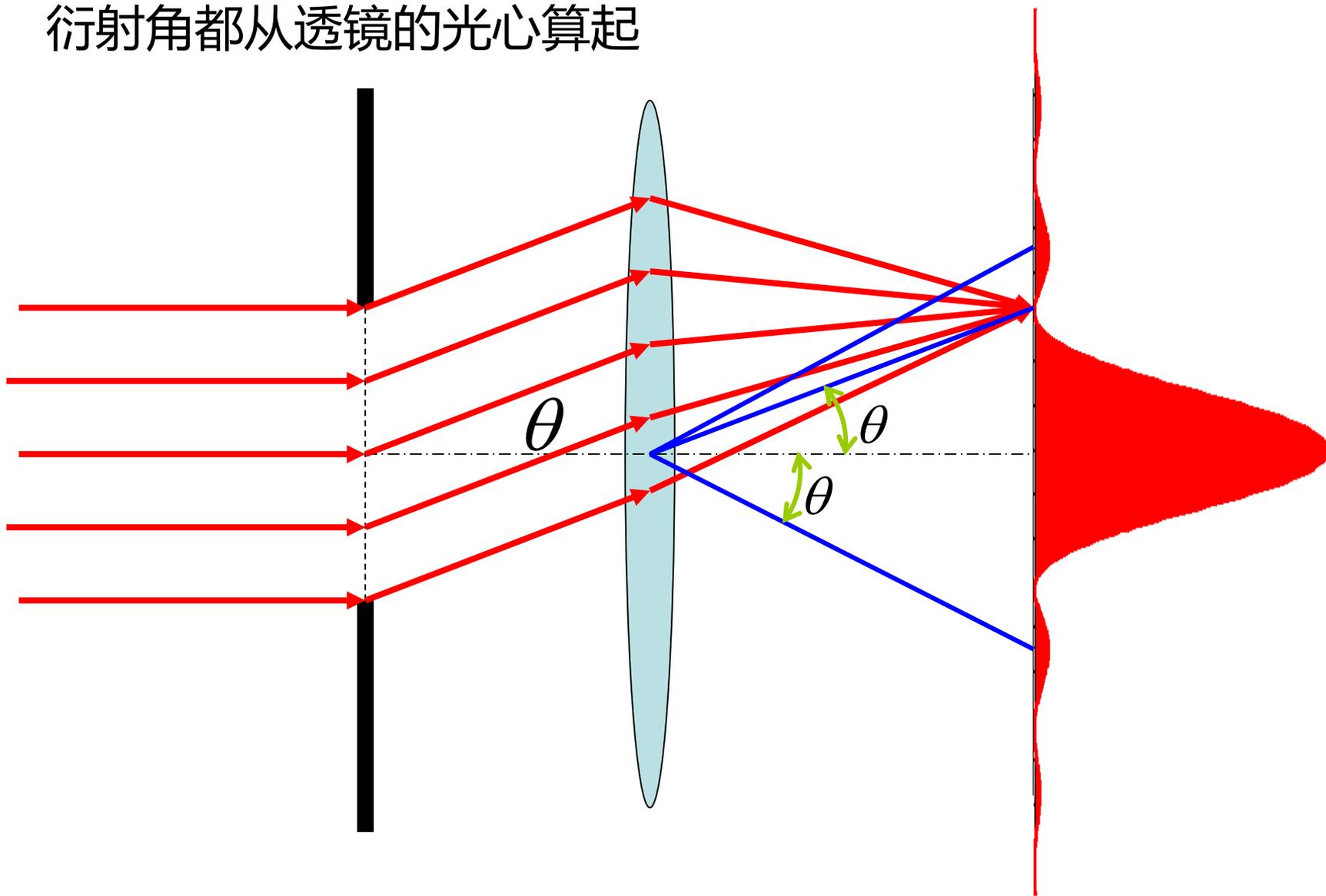
入射光与光轴不平行  
，光程差包括两部分



$$-x(\sin \theta_0 \pm \sin \theta)$$

$$u = \frac{\pi a}{\lambda} (\sin \theta_0 \pm \sin \theta)$$

衍射角都从透镜的光心算起

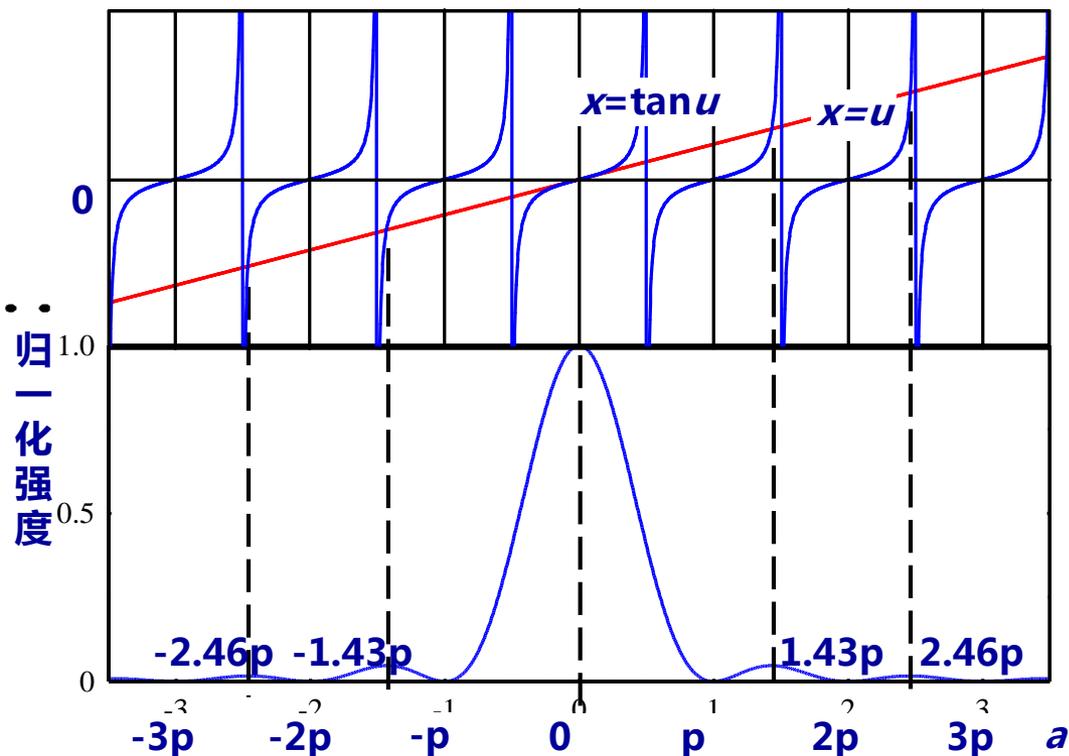


### 3. 单缝衍射因子的特点

#### 1. 极值点

**极大值**  $\left(\frac{\sin u}{u}\right)' = 0 \quad \frac{u \cos u - \sin u}{u^2} = 0 \quad \tan u = u$

$$\sin \theta = \pm 1.43 \frac{\lambda}{a},$$
$$\pm 2.46 \frac{\lambda}{a}, \pm 3.47 \frac{\lambda}{a}, \dots$$



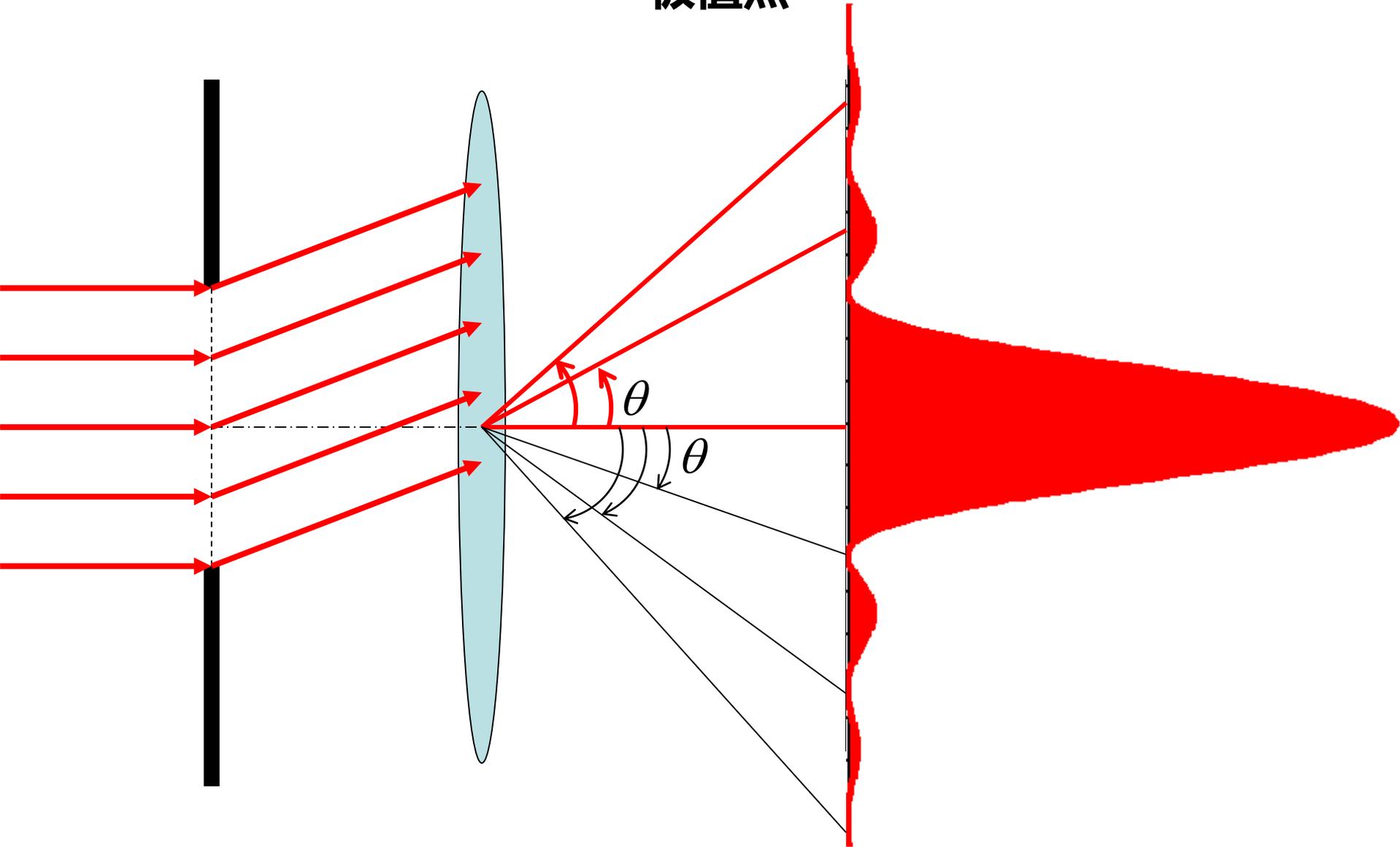
## 极小值位置

$$\tilde{U}(\theta) = \tilde{U}_0 \frac{\sin u}{u}$$

极小值  $u = \frac{\pi a}{\lambda} \sin \theta = j\pi \quad j \neq 0$

$$\sin \theta = \pm 1 \frac{\lambda}{a}, \pm 2 \frac{\lambda}{a}, \pm \dots, \pm j \frac{\lambda}{a}, \pm (j+1) \frac{\lambda}{a}, \dots$$

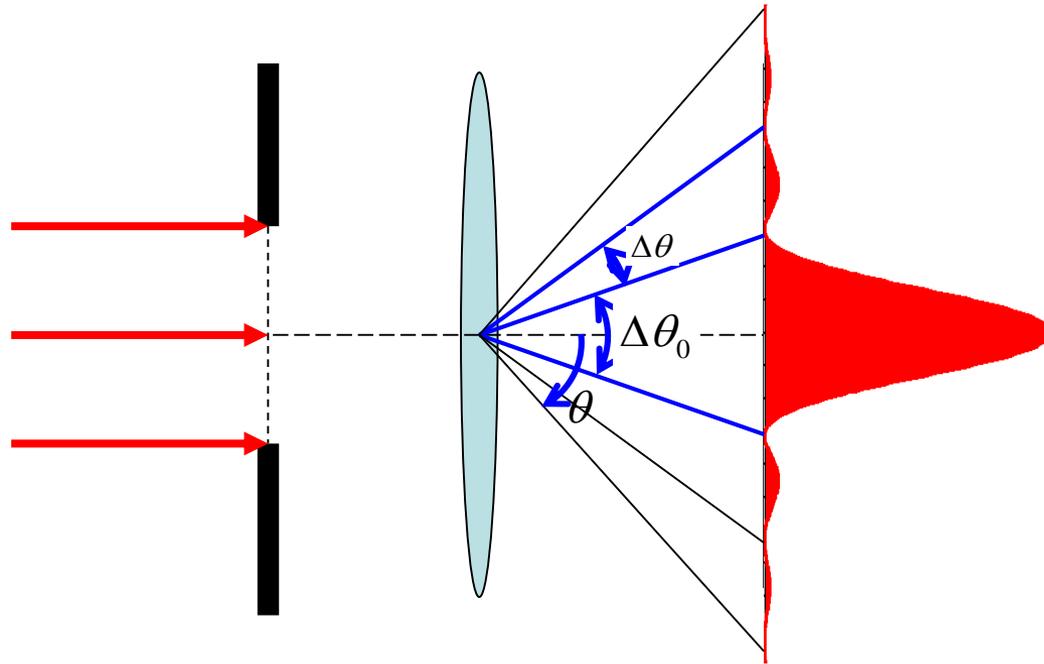
# 极值点



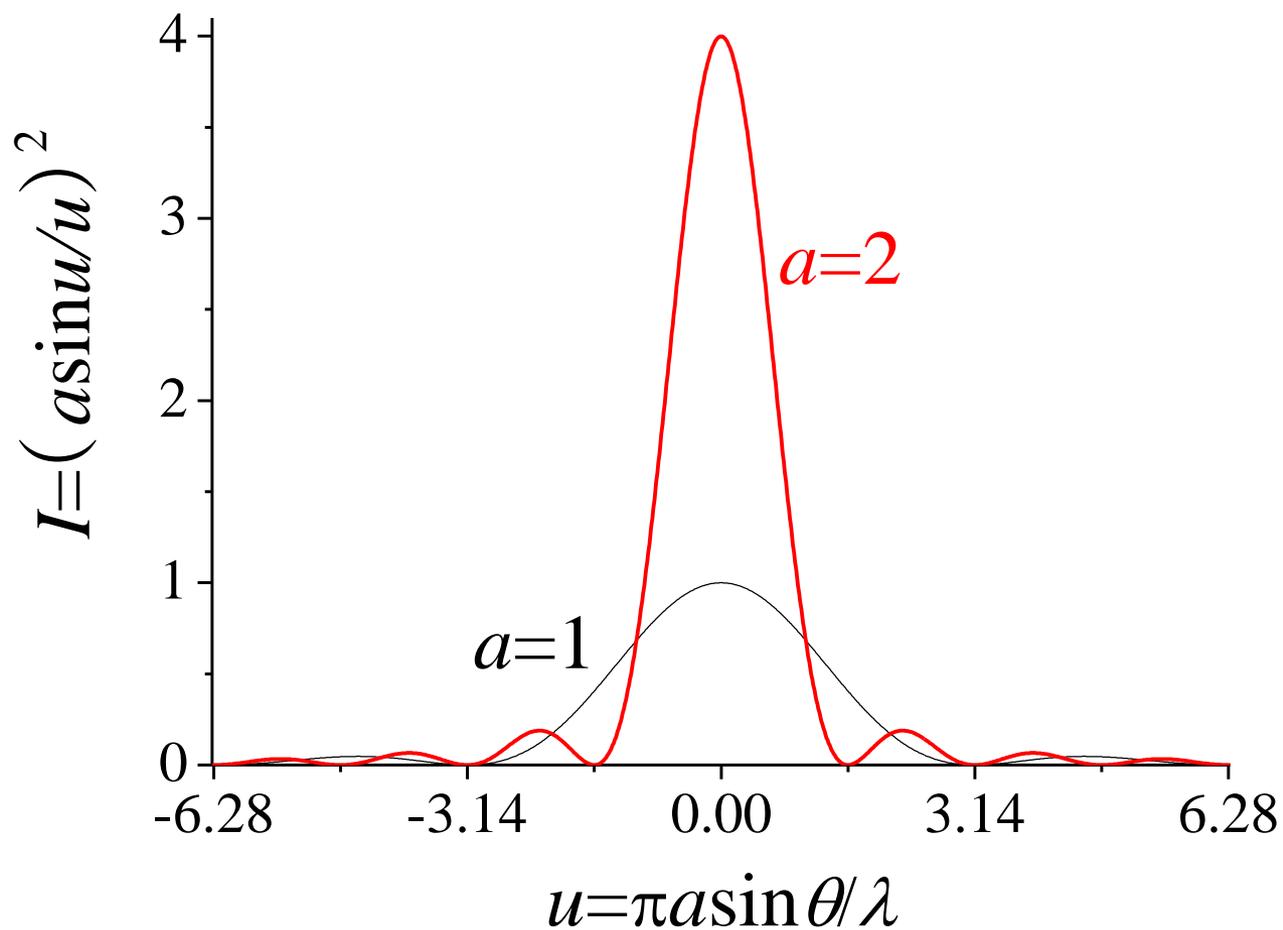
## 2. 亮条纹角宽度 ( 相邻暗条纹之间的角距离)

零级主极大  $\Delta\theta_0 = 2\frac{\lambda}{a}$

其它高级次条纹  $\Delta\theta = \frac{\lambda}{a}$  衍射的反比关系



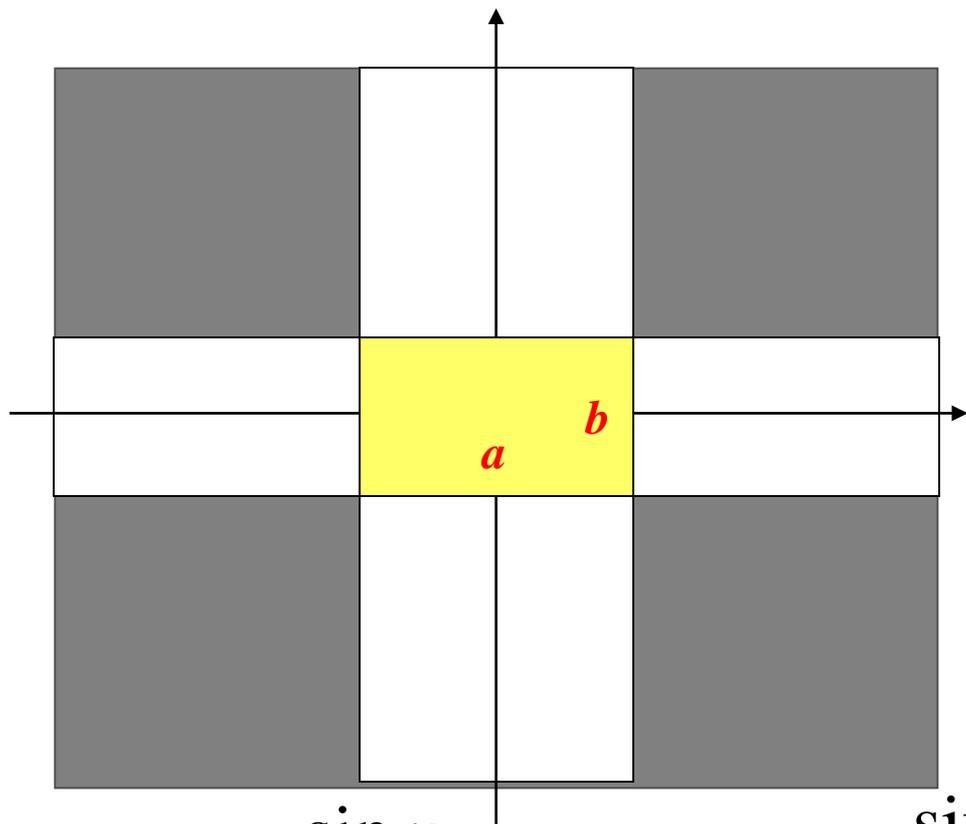
# 缝宽与光强分布



## 4. 夫琅和费矩孔衍射

- 同单缝相比，矩孔在两个相互垂直的方向上对光的传播进行限制
- 两个方向的参数是相互独立的
- 最后的结果应该是两个方向的单元衍射因子的乘积

$$\tilde{U}(x, y) = \tilde{U}(x)\tilde{U}(y)$$

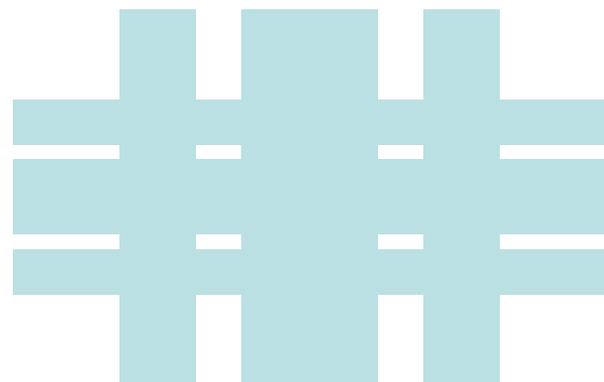


矩孔：  
两个正交狭缝的交集

矩孔衍射：  
两个正交单狭缝衍射  
的交集

$$\tilde{U}(x) = \tilde{U}_x \frac{\sin u_x}{u_x} \quad \tilde{U}(y) = \tilde{U}_y \frac{\sin u_y}{u_y}$$

$$\tilde{U}(x, y) = \tilde{U}(x)\tilde{U}(y)$$



## 矩孔衍射强度及衍射图样

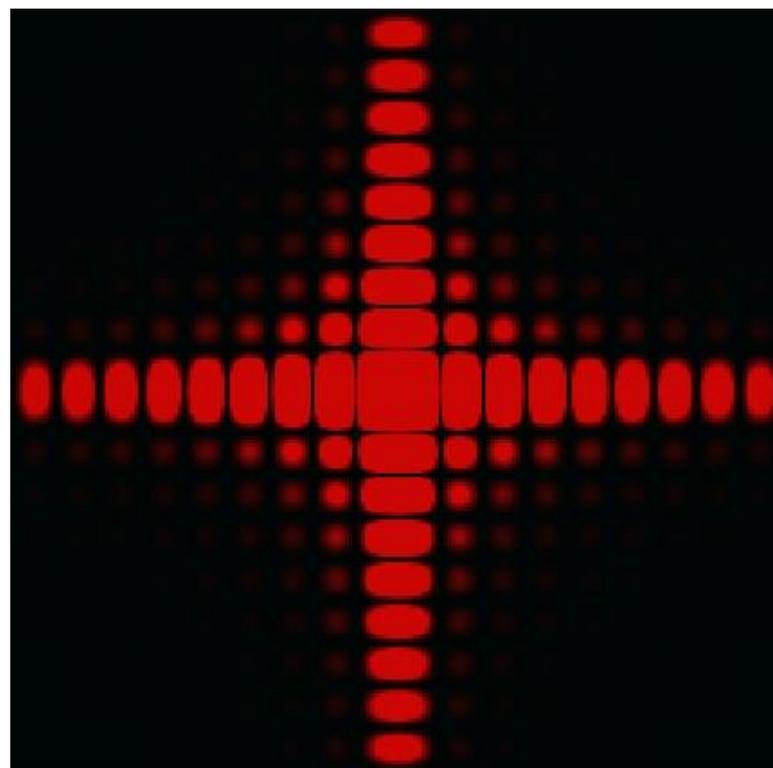
$$\tilde{U}(P) = K\tilde{U}_0(0,0)ab \frac{e^{ikr_0}}{r_0} \frac{\sin u_1}{u_1} \frac{\sin u_2}{u_2}$$

衍射强度分布

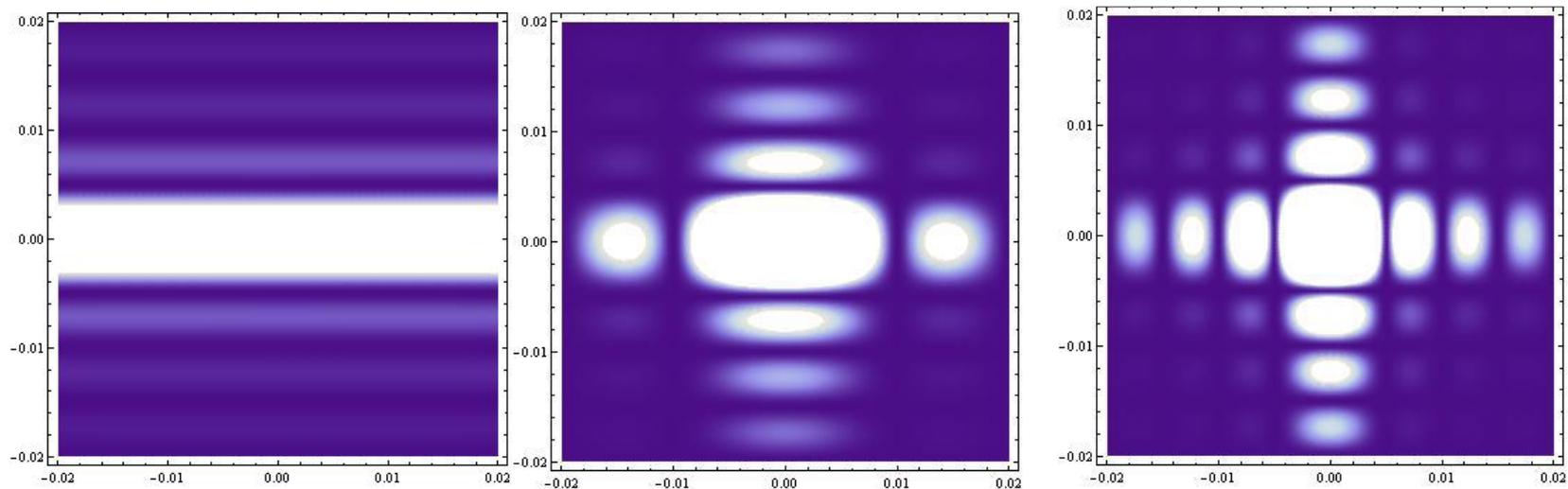
$$I(P) = I_0 \left( \frac{\sin u_1}{u_1} \right)^2 \left( \frac{\sin u_2}{u_2} \right)^2$$

$$I_0 = \left| K\tilde{U}_0(0,0)ab \frac{e^{ikr_0}}{r_0} \right|^2$$

矩孔发出的光波在F点  
产生的光强



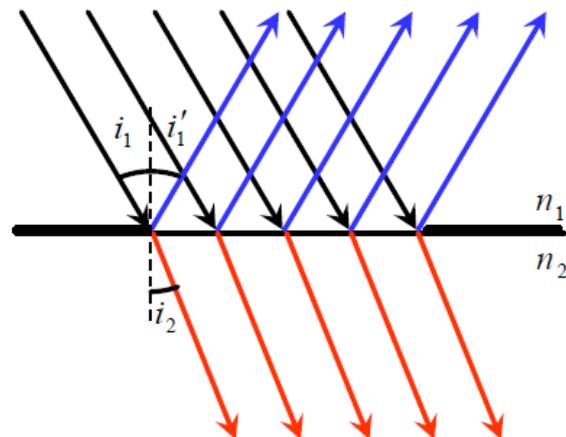
夫琅禾费矩孔衍射的图示，三图分别为矩孔长宽之比100:1, 2:1, 1:1 的情况



## 5. 衍射反比关系的意义

### i) 几何光学极限

反射波场、透射波场都是平行光斜入射的矩孔衍射，其零级主极大就是几何光学反射和透射光的方向。



### ii) 波场中的能量分布与参与相干迭加波的数目（面积）有关：

一个点源：各向同性

两个点源：某些方向出现干涉极大

.....

参与干涉的波源数目越多，出现干涉极大的条件越苛刻，能量也越集中在某个特定的方向上。

### iii) 衍射的放大作用

测量单缝或细丝的宽度或直径

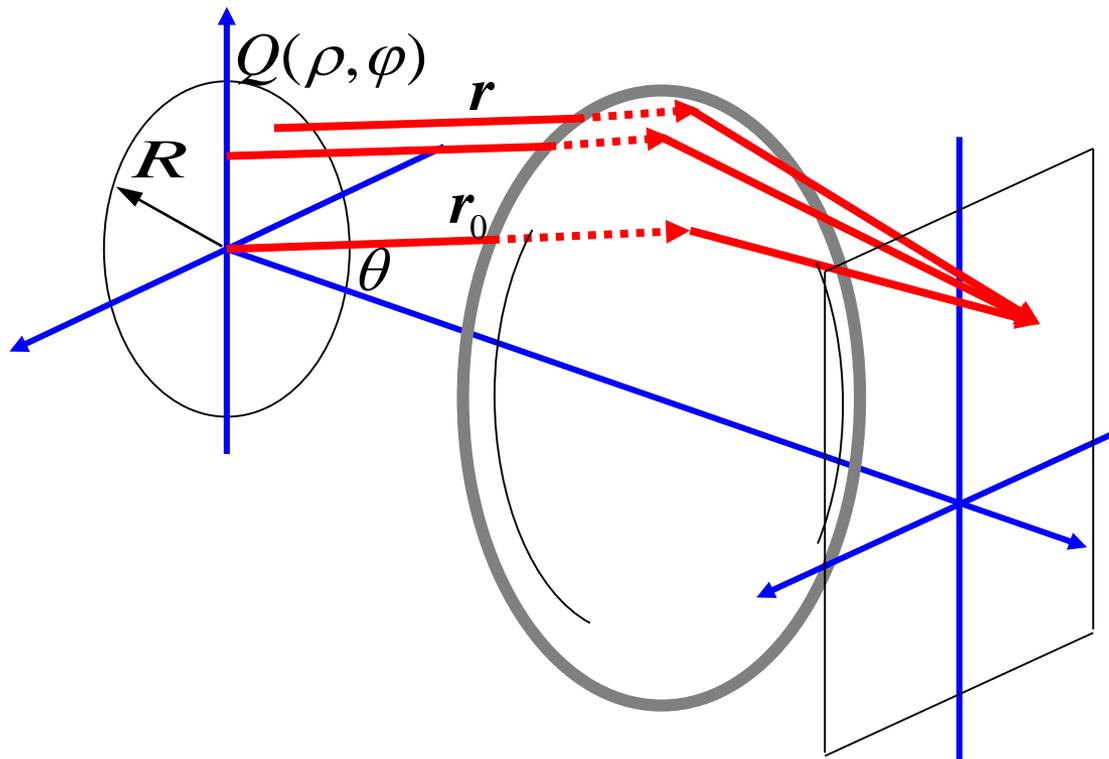
**作业 : P224-225: 1, 2, 4**

# 4-04夫琅和费 ( Fraunhofer ) 圆孔衍射和光学仪器的像分辨本领

- 1 夫琅和费 ( Fraunhofer ) 圆孔衍射
  - 2 望远镜的分辨本领, 瑞利判据
  - 3 球面波照明下像平面上接收的衍射
  - 4 显微镜的分辨本领
- ★ 提高显微镜分辨本领的方法

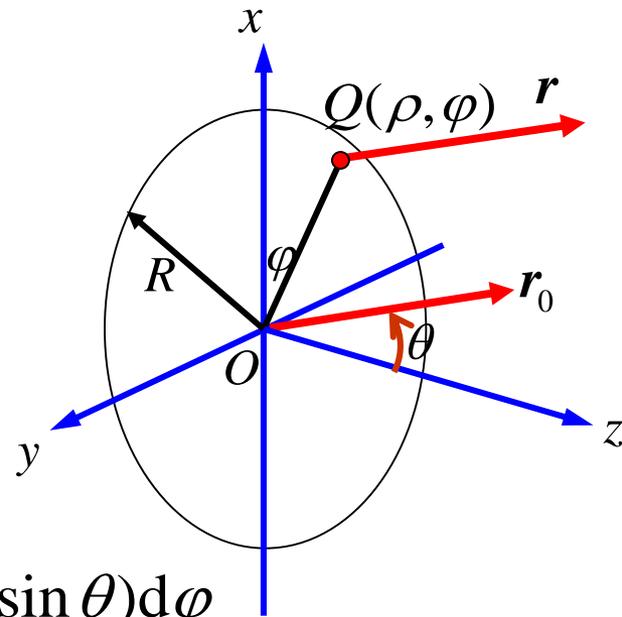
# 1. 夫琅和费 ( Fraunhofer ) 圆孔衍射

- 平行入射光，通过半径为  $R$  的圆孔，汇聚在透镜的像方焦平面上。





# 圆孔衍射的积分计算



$$\tilde{U}(P) = K \iint_{\Sigma} \tilde{U}_0(\rho, \varphi) F(\theta_0, \theta) \frac{e^{ikr_0 - ik\rho \cos \varphi \sin \theta}}{r} d\Sigma$$

$$= K \tilde{U}_0(0, 0) \frac{e^{ikr_0}}{r_0} \iint_{\Sigma} e^{-ik\rho \cos \varphi \sin \theta} \rho d\varphi d\rho$$

$$= K \tilde{U}_0(0, 0) \frac{e^{ikr_0}}{r_0} \int_0^R \rho d\rho \int_0^{2\pi} \cos\left(\frac{2\pi}{\lambda} \rho \cos \varphi \sin \theta\right) d\varphi$$

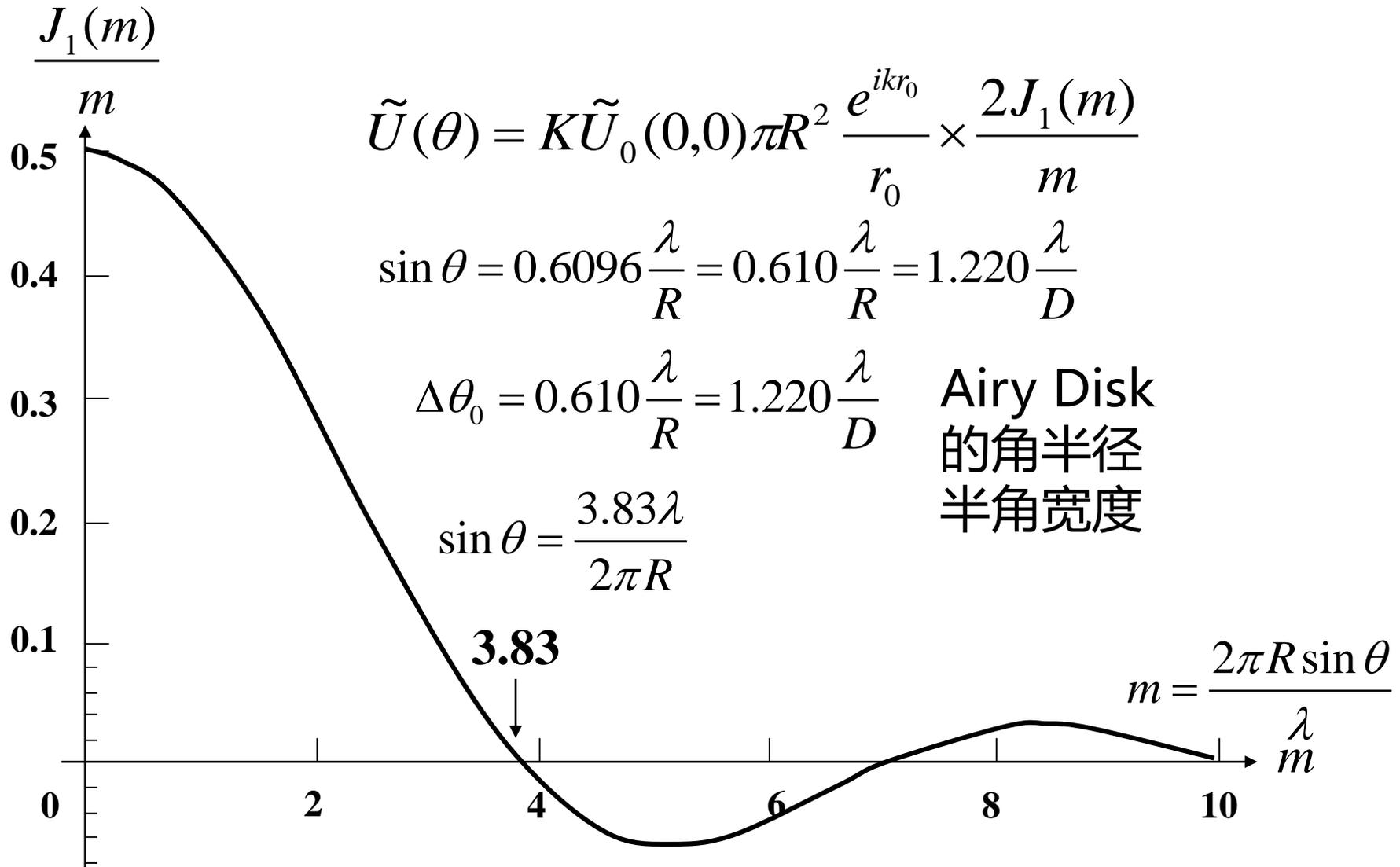
$$m = 2\pi R \sin \theta / \lambda = kR \sin \theta$$

$$\tilde{U}(P) = K \tilde{U}_0(0, 0) \frac{e^{ikr_0}}{r_0} \int_0^R \rho d\rho \int_0^{2\pi} \cos\left(m \frac{\rho}{R} \cos \varphi\right) d\varphi$$

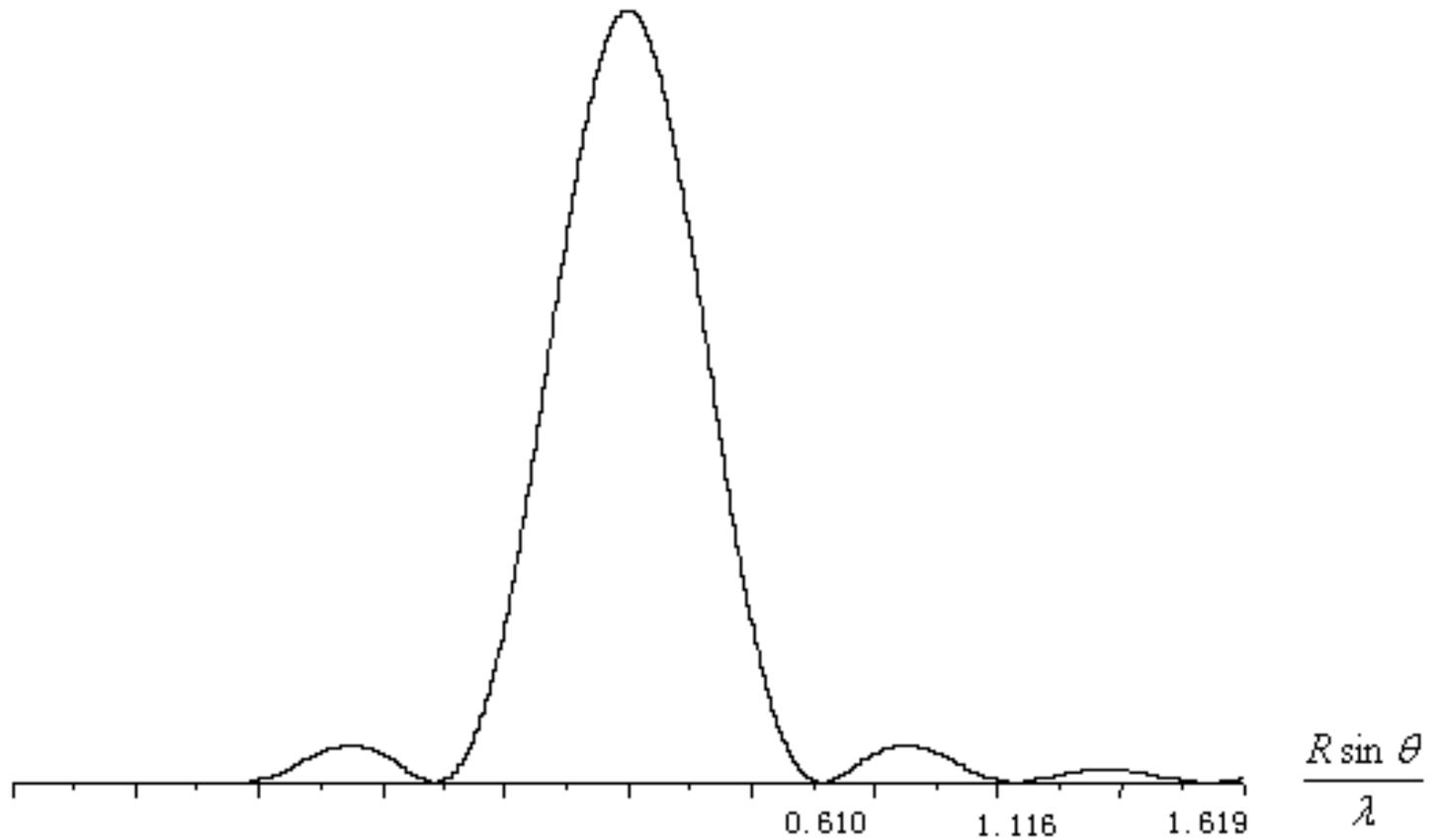
$$\tilde{U}(\theta) = K \tilde{U}_0(0, 0) \pi R^2 \frac{e^{ikr_0}}{r_0} \times \frac{2J_1(m)}{m} \quad J_1(m) : \text{一阶贝塞尔函数}$$

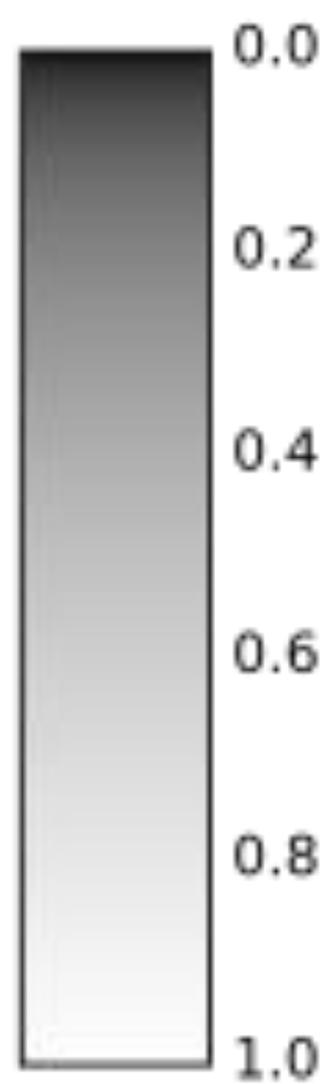
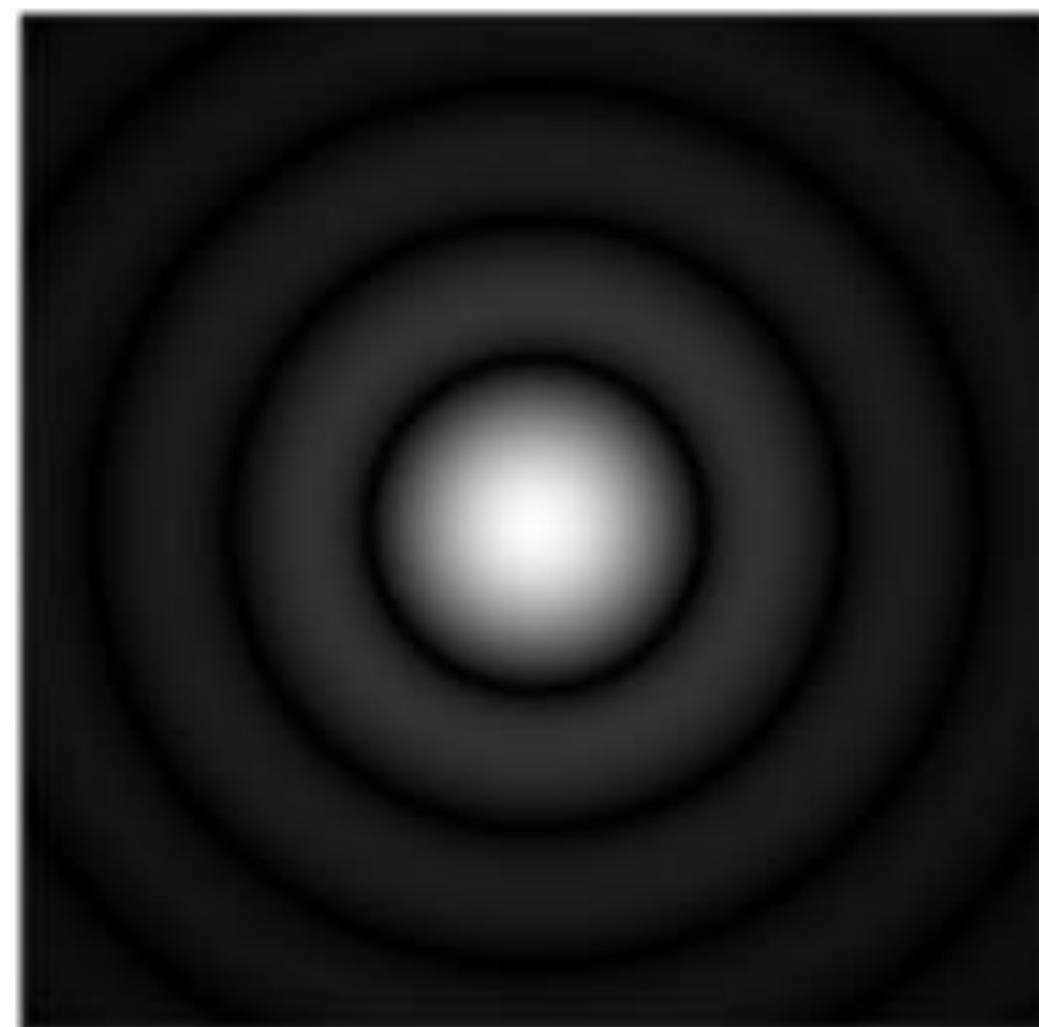
$$\frac{2J_1(m)}{m} = \sum_{k'=0}^{\infty} \frac{(-1)^{k'}}{(k'+1)!k'} \left(\frac{m}{2}\right)^{2k'} = 1 - \frac{1}{2} \left(\frac{m}{2}\right)^2 + \frac{1}{3} \left(\frac{1}{2!}\right)^2 \left(\frac{m}{2}\right)^6 - \frac{1}{4} \left(\frac{1}{3!}\right)^2 \left(\frac{m}{2}\right)^8 + \dots$$

# 复振幅分布

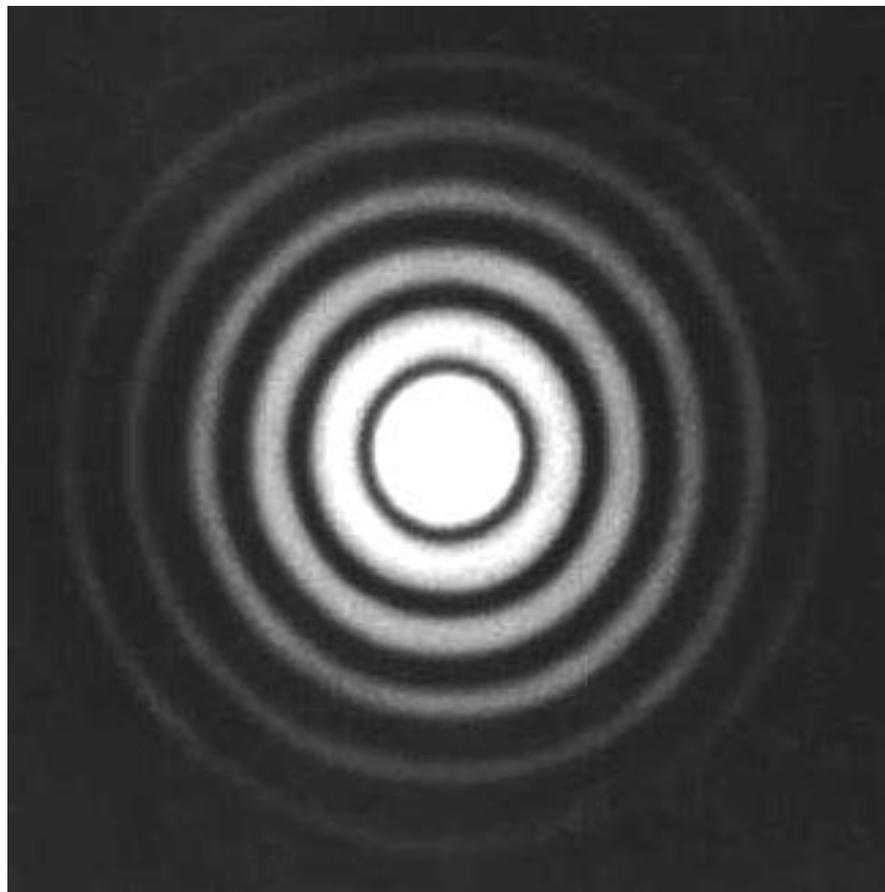


$$I(\theta) = I_0 \left[ \frac{2J_1(m)}{m} \right]^2$$





# 圆孔的夫琅禾费衍射图样

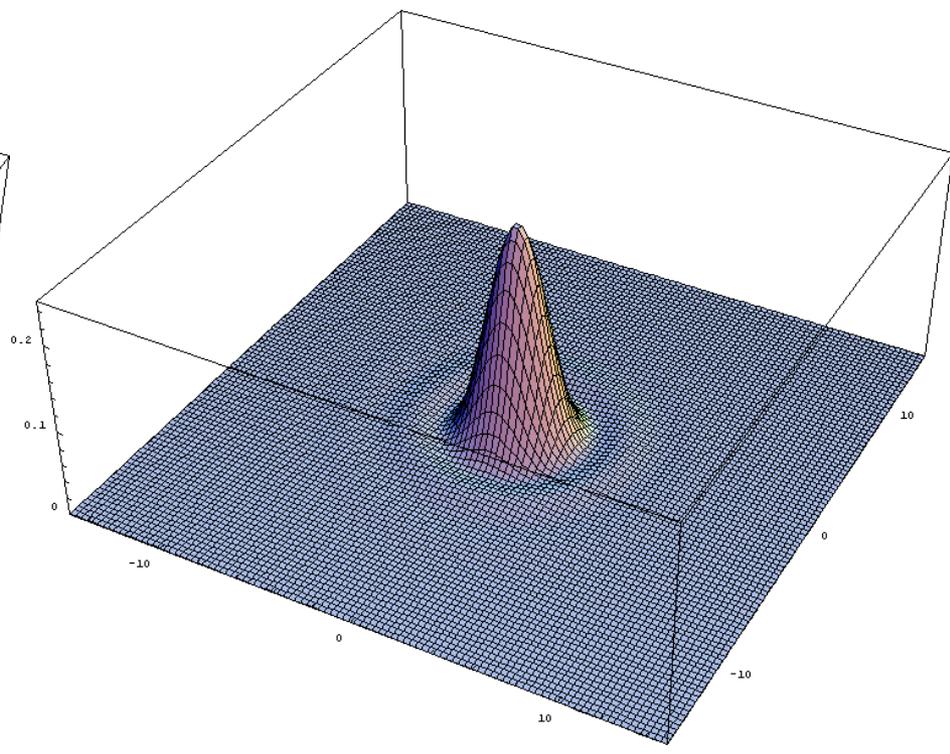
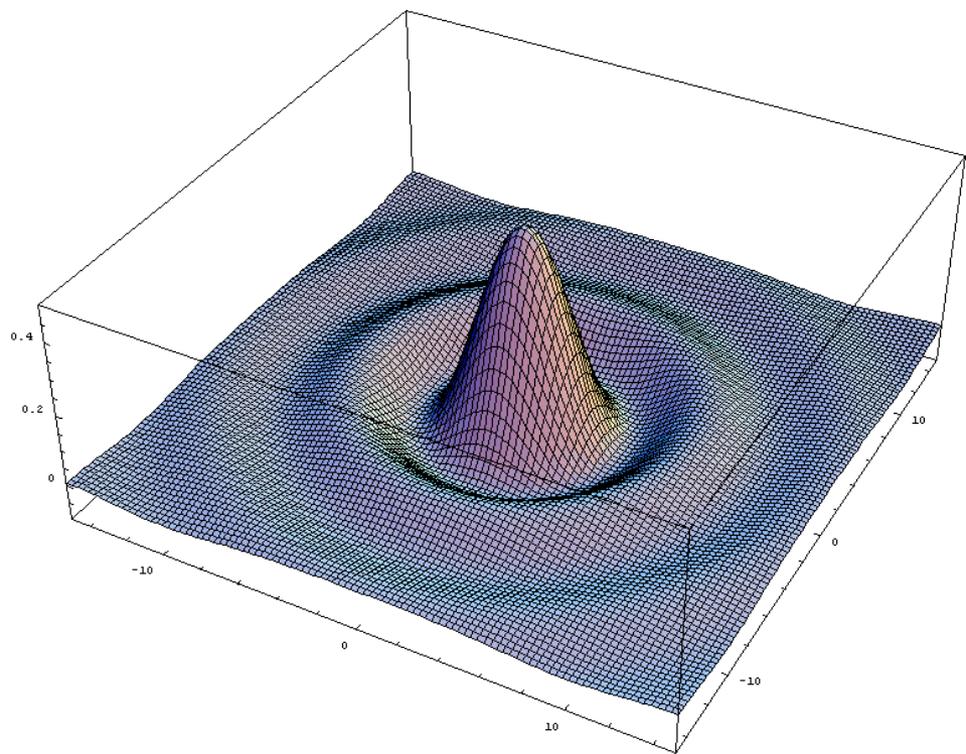


单色光



白光

# 衍射强度分布



# 衍射花样的特点

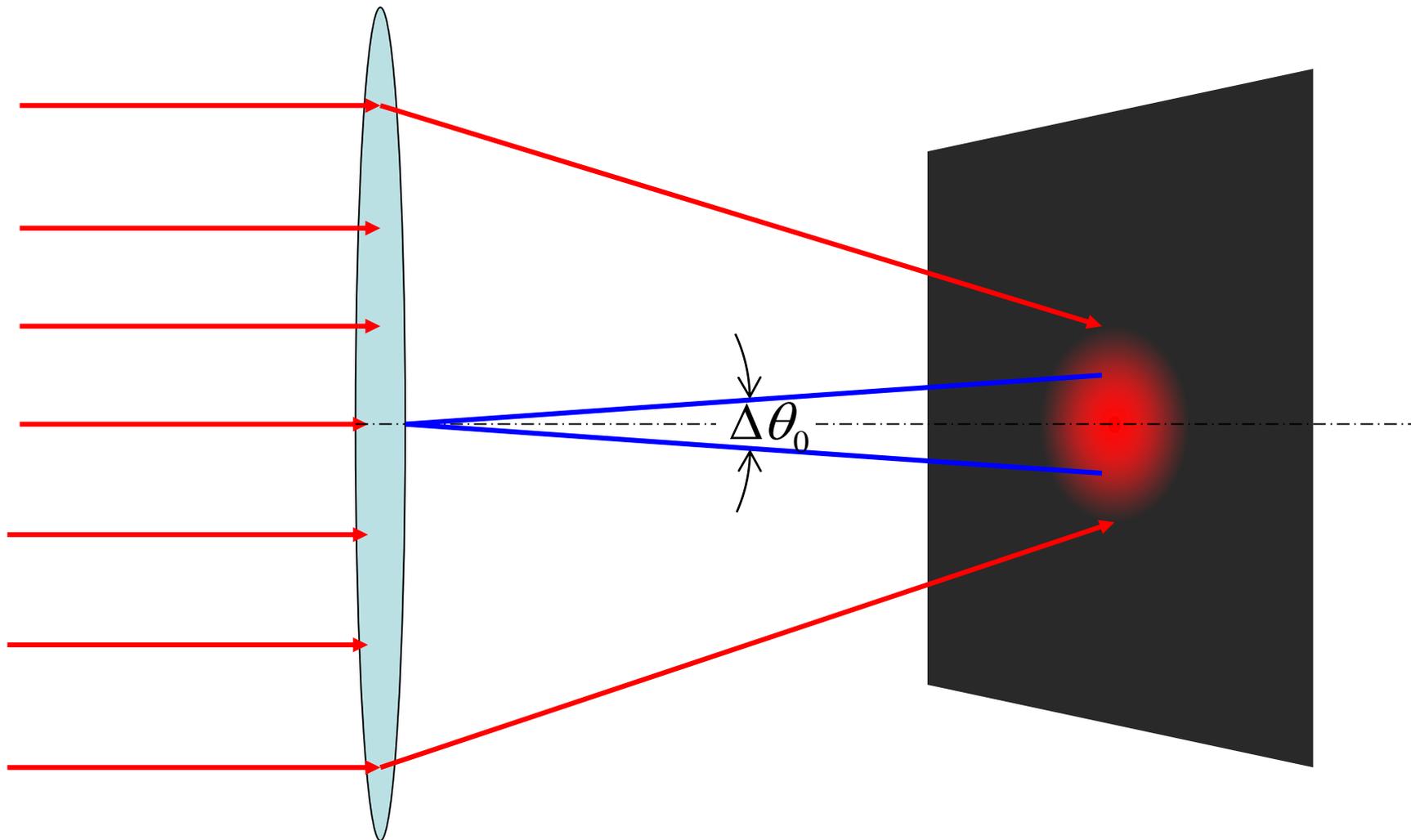
- 同心圆环，明暗交错，不等距。
- 中央主极大（零级斑）：Airy斑，占总强度的84%，  
**半角宽度  $\Delta\theta_0$**

$$\Delta\theta_0 = 0.61 \frac{\lambda}{R} = 1.22 \frac{\lambda}{D}$$

- 圆孔直径为D，透镜焦距f，则Airy斑半径 $\Delta l$

$$\Delta l = f \cdot \theta_0 = 1.22 \frac{\lambda}{D}$$

# Airy Disk 强度分布示意

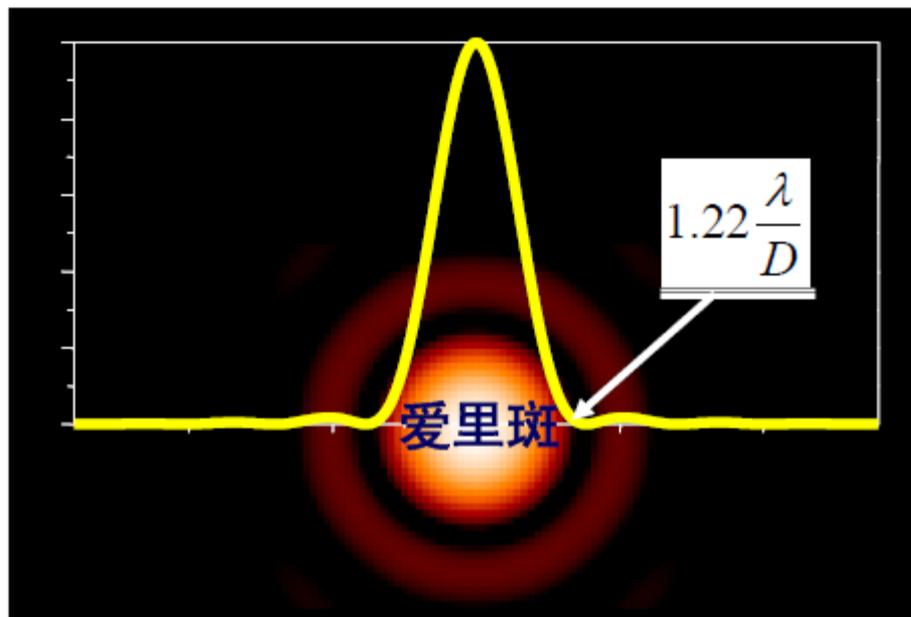


爱里斑 ( G.B. Airy , 1835 ) ( 主极大 ) :

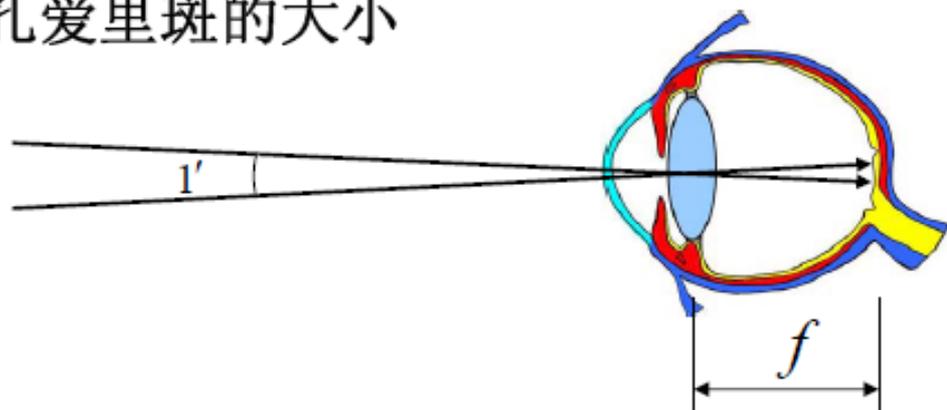
$$\text{半角宽} : \Delta\theta_0 = 0.610 \frac{\lambda}{R} = 1.220 \frac{\lambda}{D} \quad \text{反比关系}$$



George Biddell Airy



例：估算人眼瞳孔爱里斑的大小



取波长为 $0.55\mu\text{m}$ ，瞳孔直径在2-8mm之间，取2mm

$$\Delta\theta_0 = 1.22 \frac{\lambda}{D} = 3.4 \times 10^{-4} \text{ rad} = 1'$$

婴儿眼睛的焦距 $\sim 16\text{mm}$ ，成人 $\sim 24\text{mm}$ ，取 $f=20\text{mm}$

爱里斑直径为

$$D = 2f\Delta\theta_0 \approx 14\mu\text{m}$$

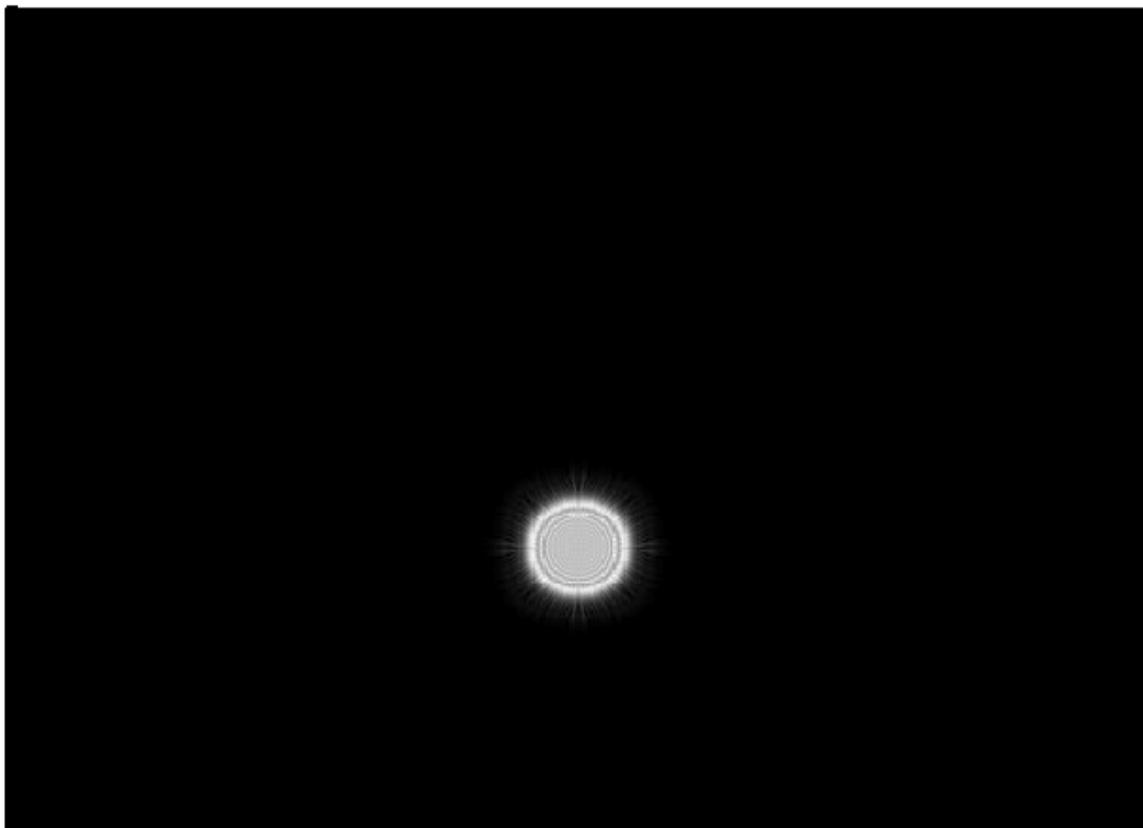
和视网膜上视觉细胞的密度正好匹配

例：1mm直径HeNe激光的衍射发射角

$$\Delta\theta_0 = 1.22 \frac{\lambda}{D} = 7.7 \times 10^{-4} \text{ rad} = 2.7'$$

10km处的光斑直径：**D=14.4m**

绝对平行的光是不存在的！

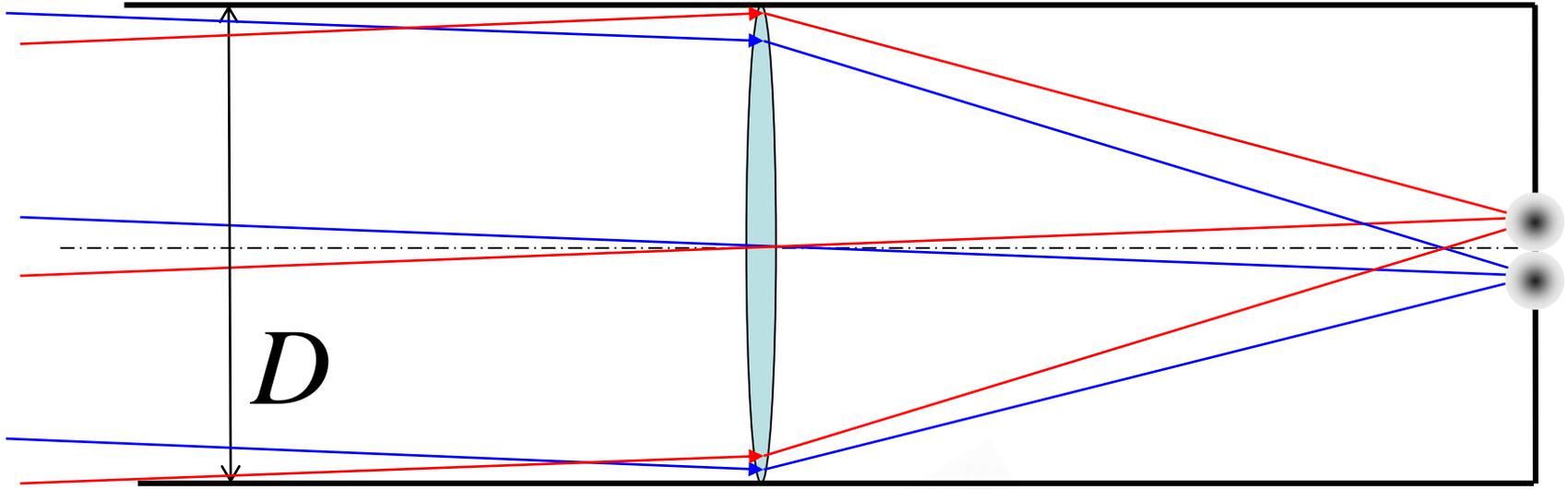


- 圆孔衍射图样：圆孔直径0.5micron，波长0.6micron (red-light)，距离0.1cm~1cm in steps of 0.1cm.
- 图样从 Fresnel region 变为 Fraunhofer region，Airy pattern 显示.

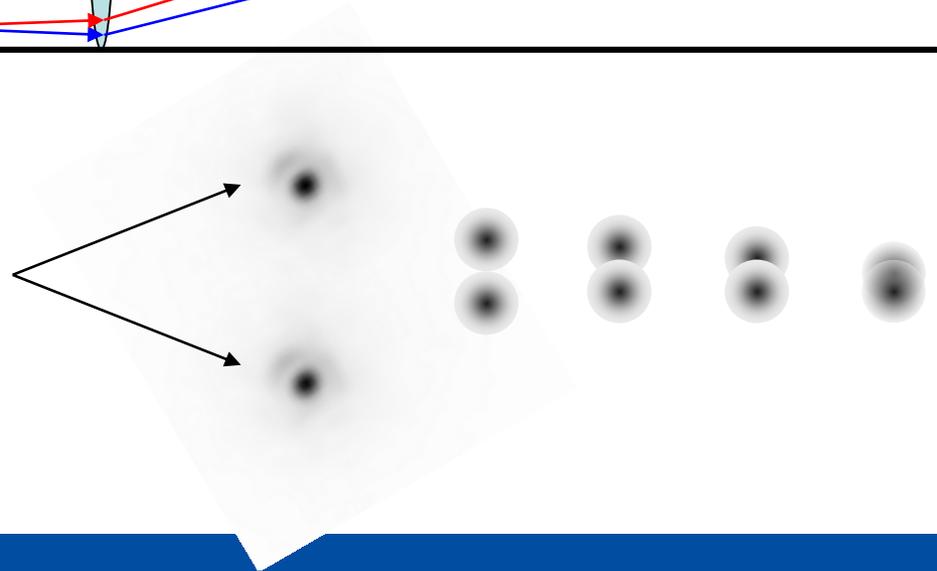
## 2. 望远镜的分辨本领, 瑞利判据

- 仰望星空，为什么它们看起来大小差不多
- 平行光经透镜成象，由于衍射效应，总有一个Airy斑，而不是一个几何点。
- 两束光，则有两个Airy斑。
- 两个物所成的Airy斑如靠得很近，可能无法分辨是一个还是两个。
- 采用Rayleigh判据：**两光斑的角距离恰等于一个光斑的半角宽度时，为可以分辨的最小极限。**

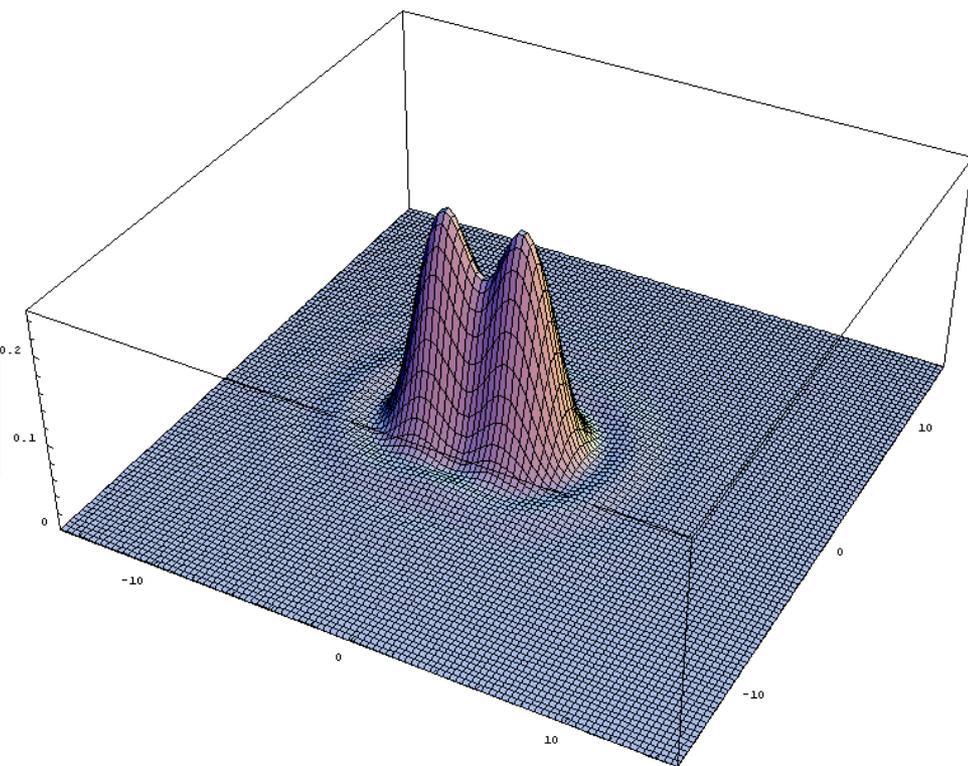
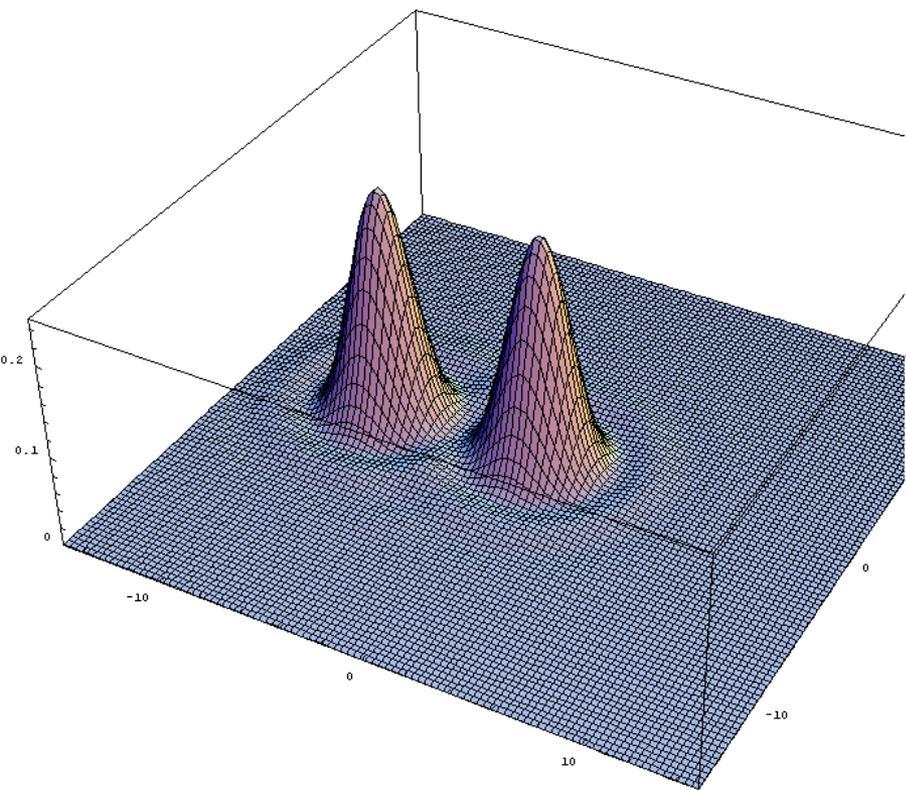
# 望远镜成像



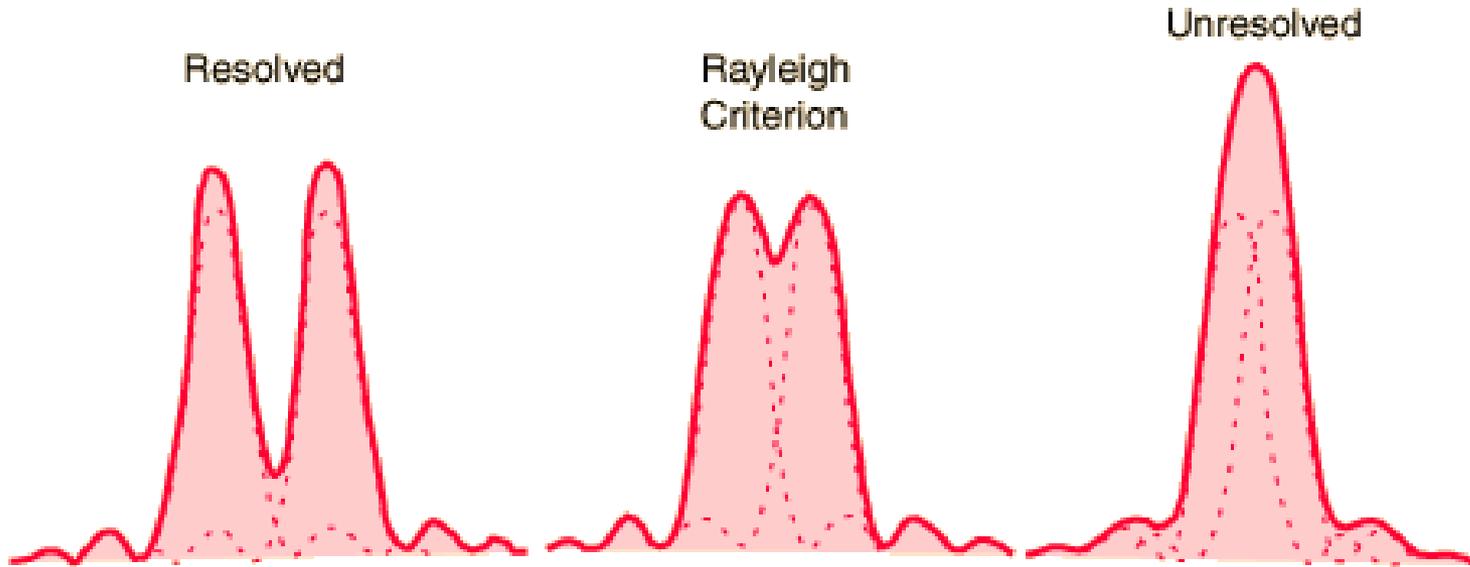
**Airy Disk**



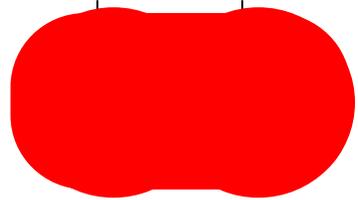
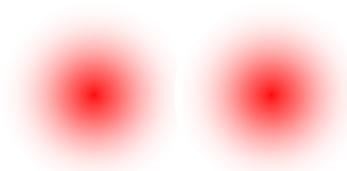
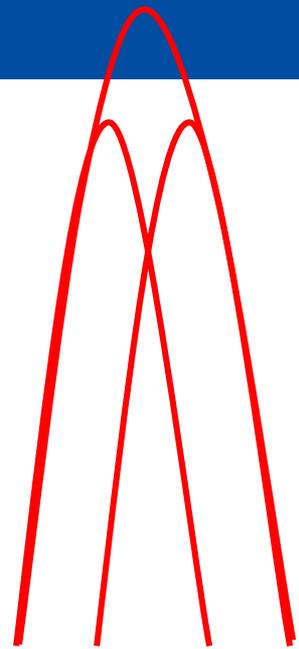
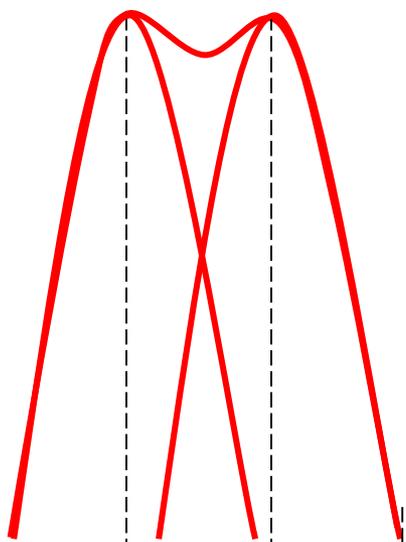
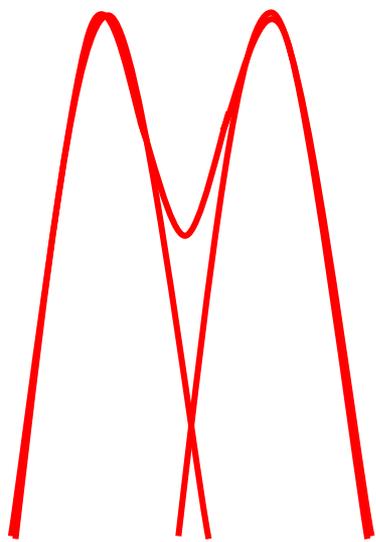
# 可分辨极限



# The Rayleigh Criterion



**The Rayleigh Criterion**



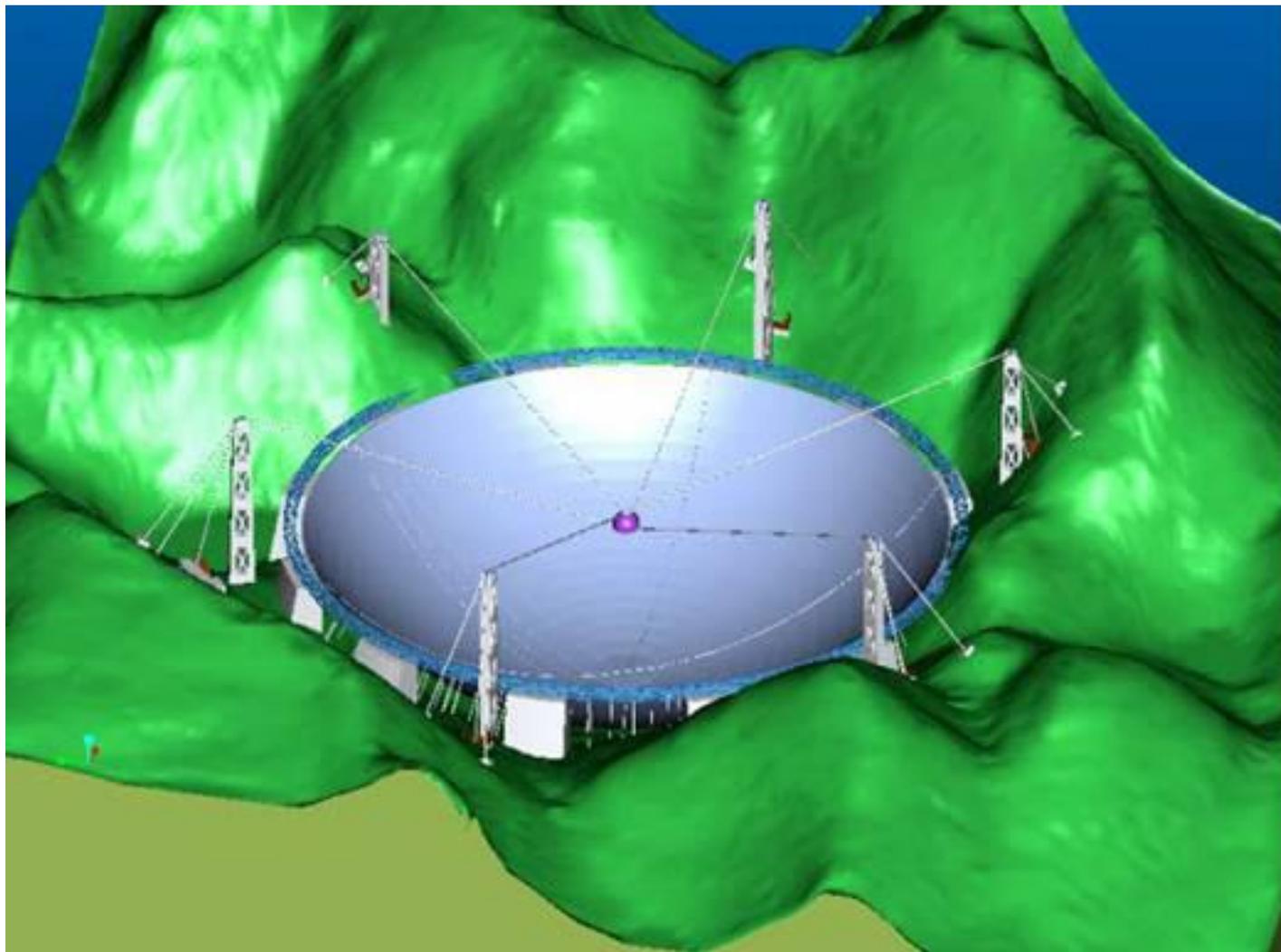
Rayleigh(瑞利)判据

恰好可以分辨

$$\delta\theta_m = \Delta\theta_0 = 1.22 \frac{\lambda}{D}$$

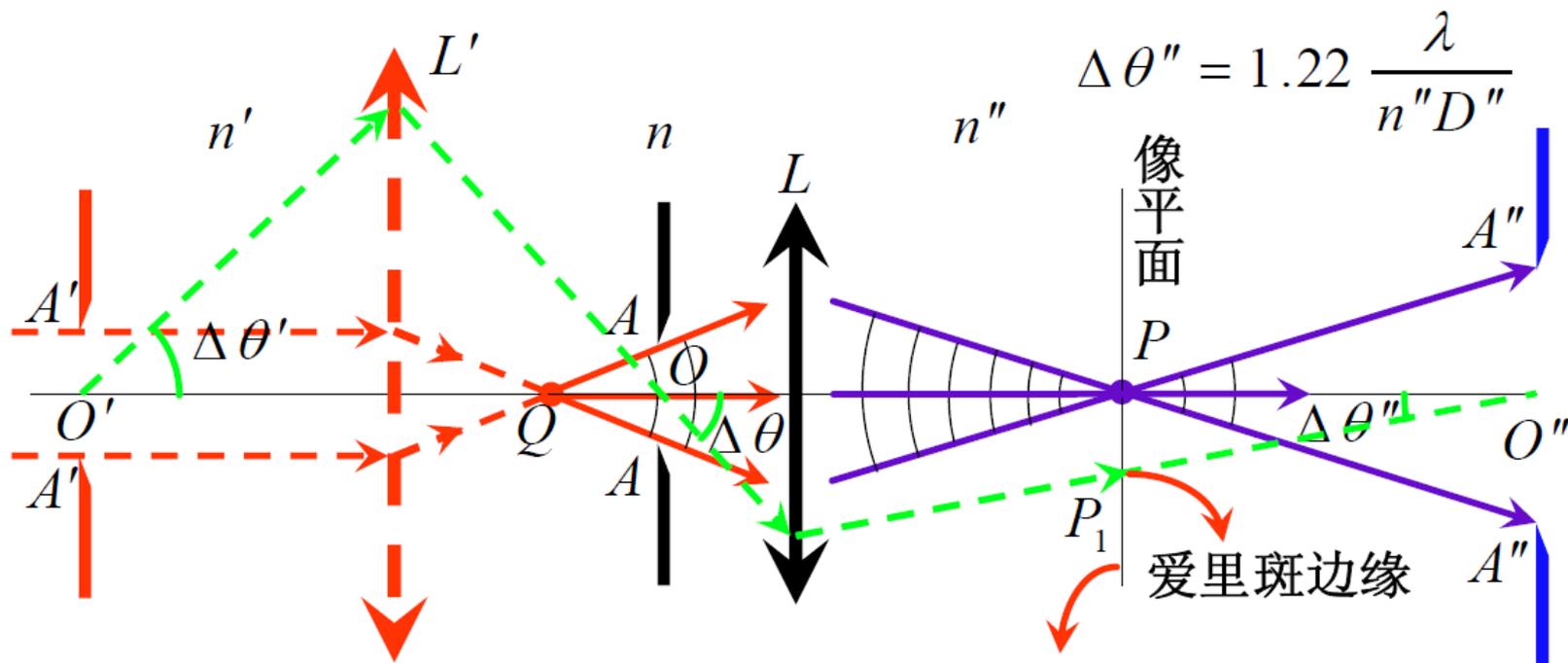


# 山体式射电天文望远镜 FAST





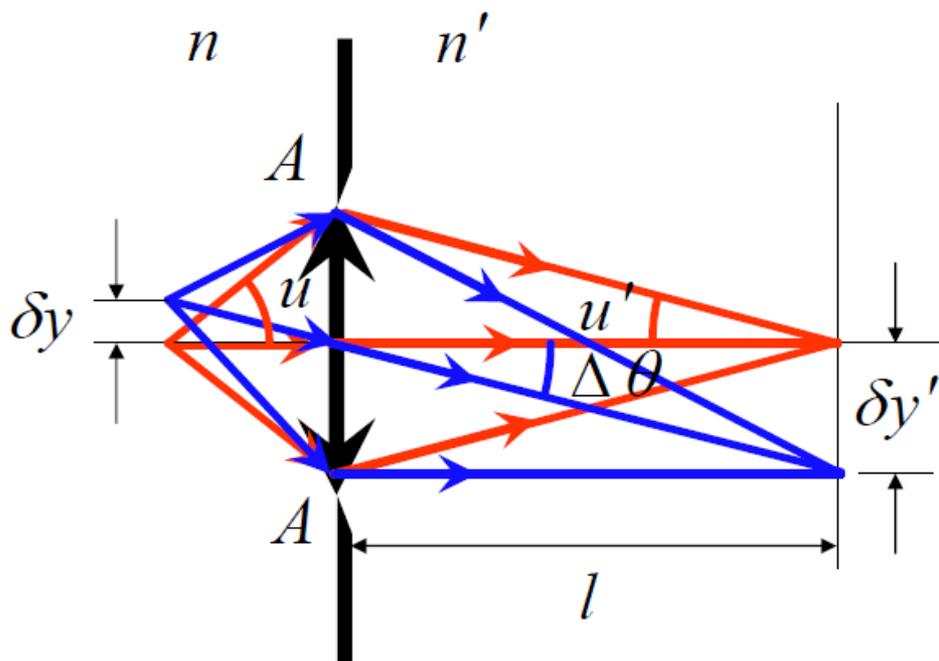
## 4.球面波照明下像平面上接收的是光瞳的夫琅和费衍射



$L'-L$ 联合光具组的焦平面，接收的是 $A'-A'$ 对联合光具组的衍射

$$nD \Delta \theta = n' D' \Delta \theta' = n'' D'' \Delta \theta'' = 1.22 \lambda$$

## 4.显微镜的分辨本领



物在光具组的焦点附近，  
同时也是齐明点。  
光瞳就是物镜的边框

$$\Delta \theta_m = \frac{1.22 \lambda}{n' D}$$

利用阿贝正弦条件

$$n \delta y \sin u = n' \delta y' \sin u'$$

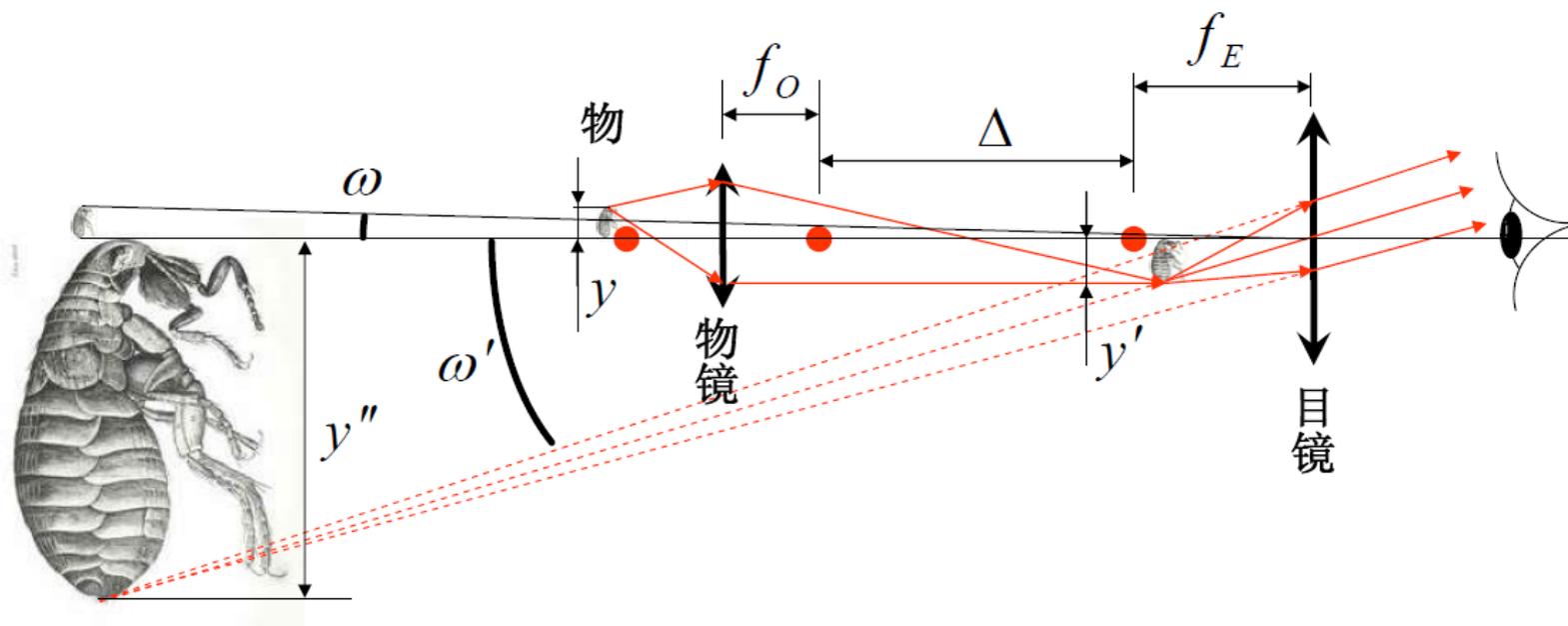
$$\sin u' \approx u' = \frac{D/2}{l}$$

$$\delta y_m = \frac{n' \frac{D/2}{l} \Delta \theta_m l}{n \sin u} = \frac{0.61 \lambda}{n \sin u}$$

$n \sin u$  数值孔径，  
**Numerical Aperture, N.A.**

## 目镜的选择：

使总放大率能将仪器的最小分辨距离放大到人眼明视距离处的最小分辨距离**0.1mm**



有效放大率：

$$M_m = \frac{\delta y_e}{\delta y_m}$$

例：显微镜数值孔径**N.A.=1.0**，估算有效放大率

取人眼最灵敏的**550nm**绿光：

$$\delta y_m \approx \frac{0.61}{1.0} \times 550nm = 335.5nm$$

$$M_m = \frac{\delta y_e}{\delta y_m} \approx 300$$

## ★ 提高显微镜分辨本领的方法—1.提高NA

油浸:

$$\delta y_m = \frac{0.61\lambda}{n \sin u}$$

远场光学显微镜的分辨极限:

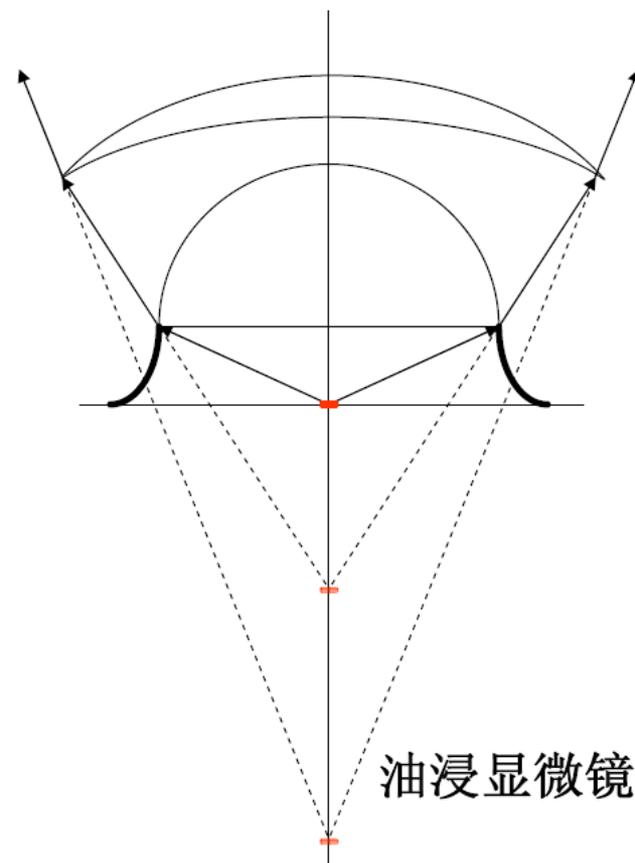
$$\delta y_m \approx \frac{0.61\lambda}{1.5} = 0.4\lambda$$

在可见波段:  $\lambda \sim 400nm$

$$\delta y_m \approx 160nm$$

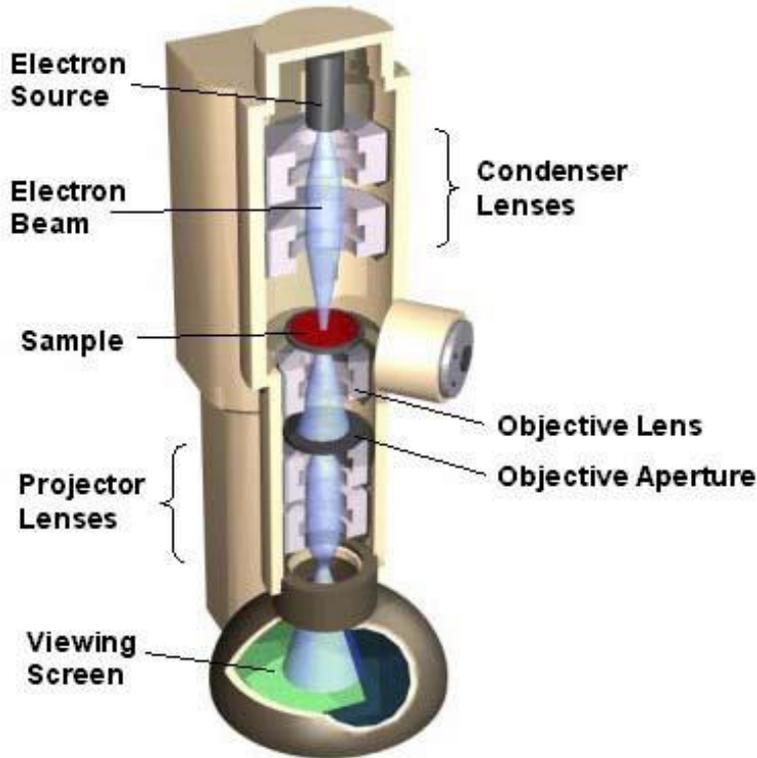
有效放大率:

$$M_m = \frac{\delta y_e}{\delta y_m} \sim 625$$



# ★ 提高显微镜分辨本领的方法

## 2. 减小波长：紫外，X射线显微镜，电子显微镜



电子显微镜的分辨本领:

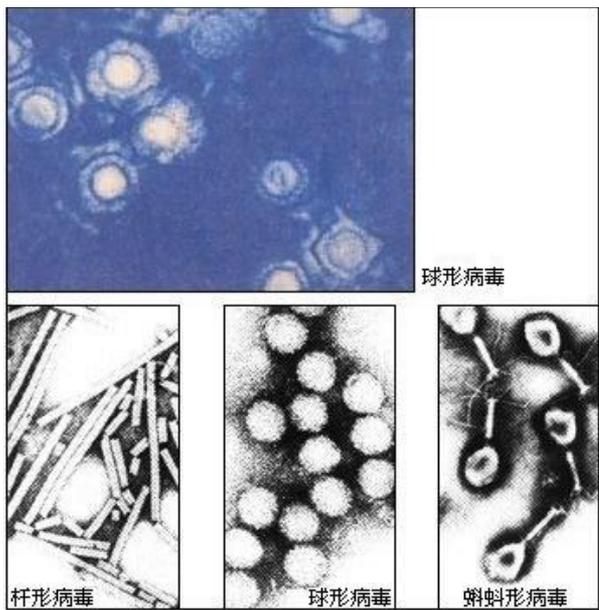
电子束发散角较小,  $u \sim 0.16 \text{ rad}$

$$\delta y_m^E \approx 0.61 \frac{\lambda_e}{N.A.} \approx 0.61 \frac{\lambda_e}{0.16} \approx 4\lambda_e$$

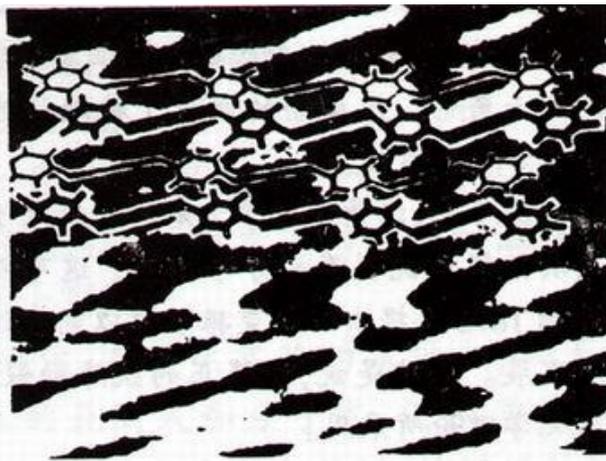
电子波长取决于电子的加速电压:

$$\frac{1}{2} \frac{p^2}{m} = eV$$
$$\lambda_e = \frac{h}{p} = \frac{h}{\sqrt{2meV}}$$

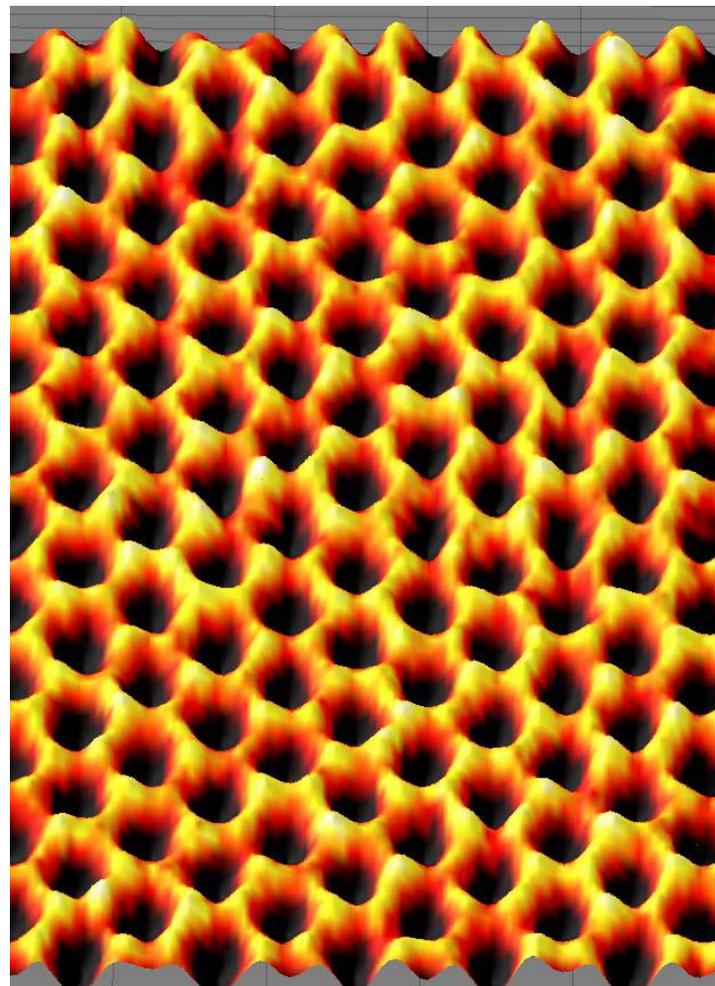
加速电压	电子波长	分辨本领	有效放大率
$10^4 \text{V}$	$1.2 \times 10^{-2} \text{nm}$	$5 \times 10^{-2} \text{nm}$	$2 \times 10^6$
$10^5 \text{V}$	$3.7 \times 10^{-3} \text{nm}$	$1.5 \times 10^{-2} \text{nm}$	$6 \times 10^6$



电子显微镜下的病毒照片



苯分子照片



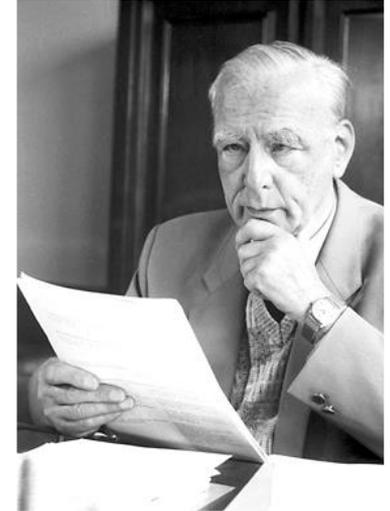
石墨烯的电镜图



**Max Knoll**



**1931 *Ruska and Knoll*  
built the first TEM**

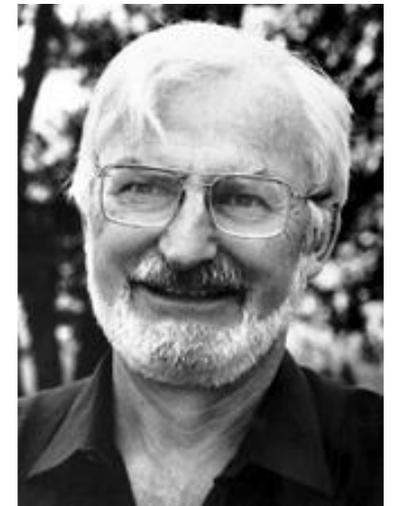


**Ernst Ruska**

The Nobel Prize in Physics 1986 was divided, one half awarded to Ernst Ruska *"for his fundamental work in electron optics, and for the design of the first electron microscope"*, the other half jointly to Gerd Binnig and Heinrich Rohrer *"for their design of the scanning tunneling microscope"*.



**Gerd Binnig**

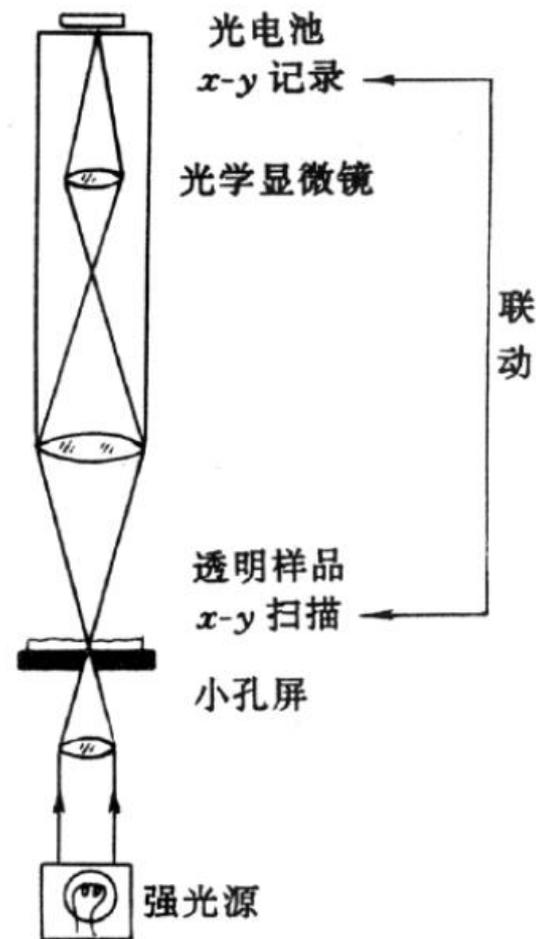


**Heinrich Rohrer**

## ★ 提高显微镜分辨本领的方法— 3. 扫描近场光学显微镜(SNOM)

Synge的设想：

- ① 在不透明的平板或薄膜上，制备出一个近10nm的小孔，置于生物样品切片的正下方，两者间隔近10nm；
- ② 入射光通过小孔照明样品，透过样品的光被显微镜聚焦到光电探测器上。
- ③ 保持入射光强度不变，在x-y方向上以10nm的步距扫描样品，探测器同步记录光强。



(phil.Mag.1928 vol.6 ,p.356)

## SNOM发展历史

1928年，Synge 提出设想；

1972年，Eric Ash等在微波波段实现近场显微；

1984年，Pohl等研制成功第一台扫描近场光学显微

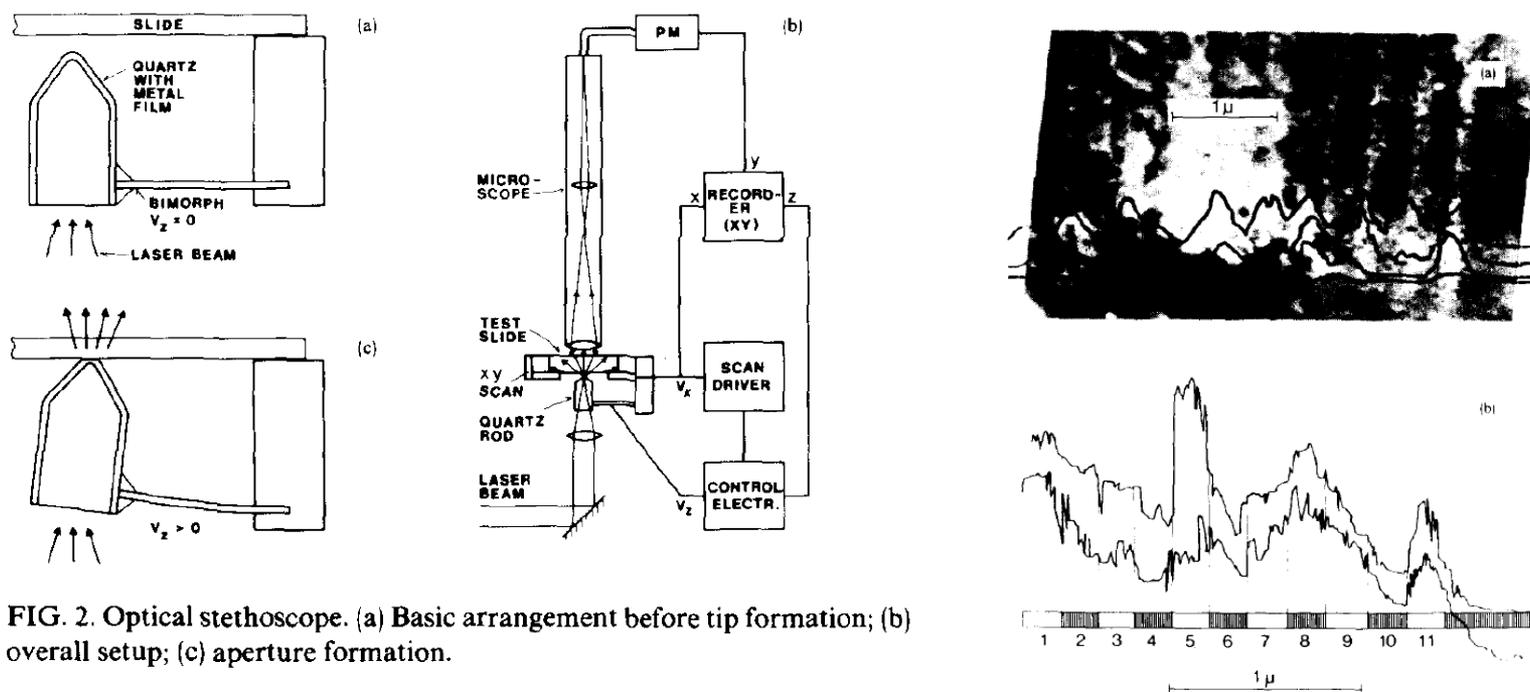


FIG. 2. Optical stethoscope. (a) Basic arrangement before tip formation; (b) overall setup; (c) aperture formation.

**D. W. Pohl, W. Denk, and M. Lanz, *Appl. Phys. Lett.* 44 (7), 651, 1984**

## Beyond diffraction limit:

### 1. Spatial difference (structured excitation/collection):

NSOM & nanostructured; SIM; STED & related

### 2. Temporal/ Lifetime difference: PALM; STORM; FLIM...

### 3. Statistics: SOFI;...

## The Nobel Prize in Chemistry 2014



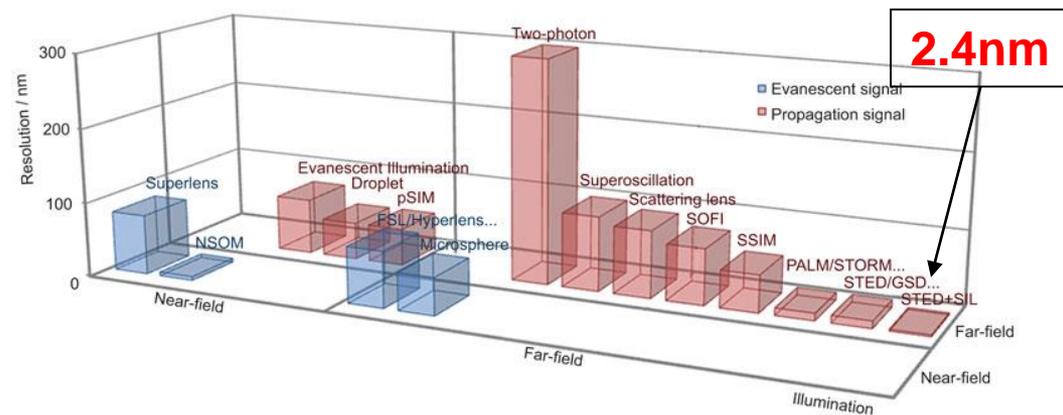
Photo: A. Mahmoud  
Eric Betzig



Photo: A. Mahmoud  
Stefan W. Hell



Photo: A. Mahmoud  
William E. Moerner



[http://www.nobelprize.org/nobel\\_prizes/chemistry/laureates/2014/](http://www.nobelprize.org/nobel_prizes/chemistry/laureates/2014/)

X. Hao, C. Kuang, Z.i Gu, Y. Wang, S. Li, Y. Ku, Y. Li, J. Ge and X. Liu,  
*From microscopy to nanoscopy via visible light*  
Light: Science & Applications 2, e108 (2013)

**作业 : P235: 1, 3, 4, 5**