

4-02费涅耳(Fresnel)圆孔衍射和圆屏衍射

- 1 实验现象
- 2 半波带法
- 3 矢量图解法
- 4 费涅耳波带片

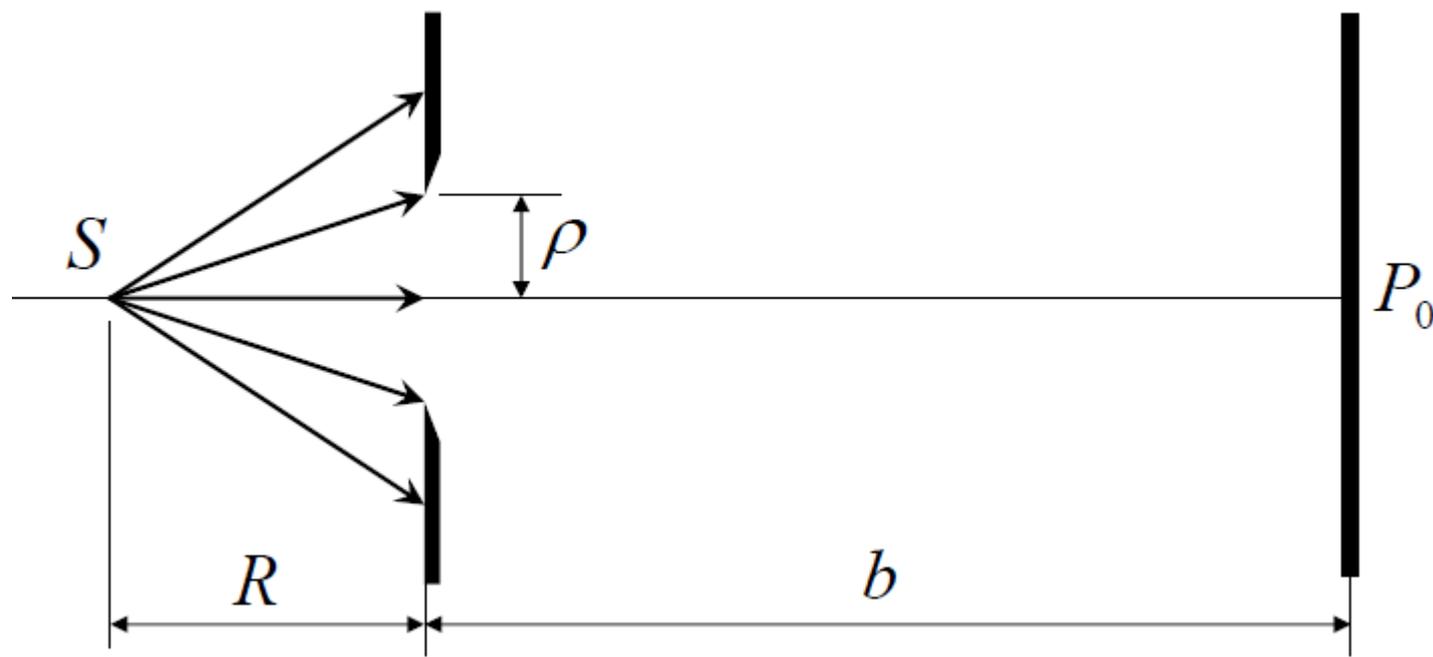
1. 实验现象

一般参数：

圆孔半径： $\rho \sim 1\text{mm}$

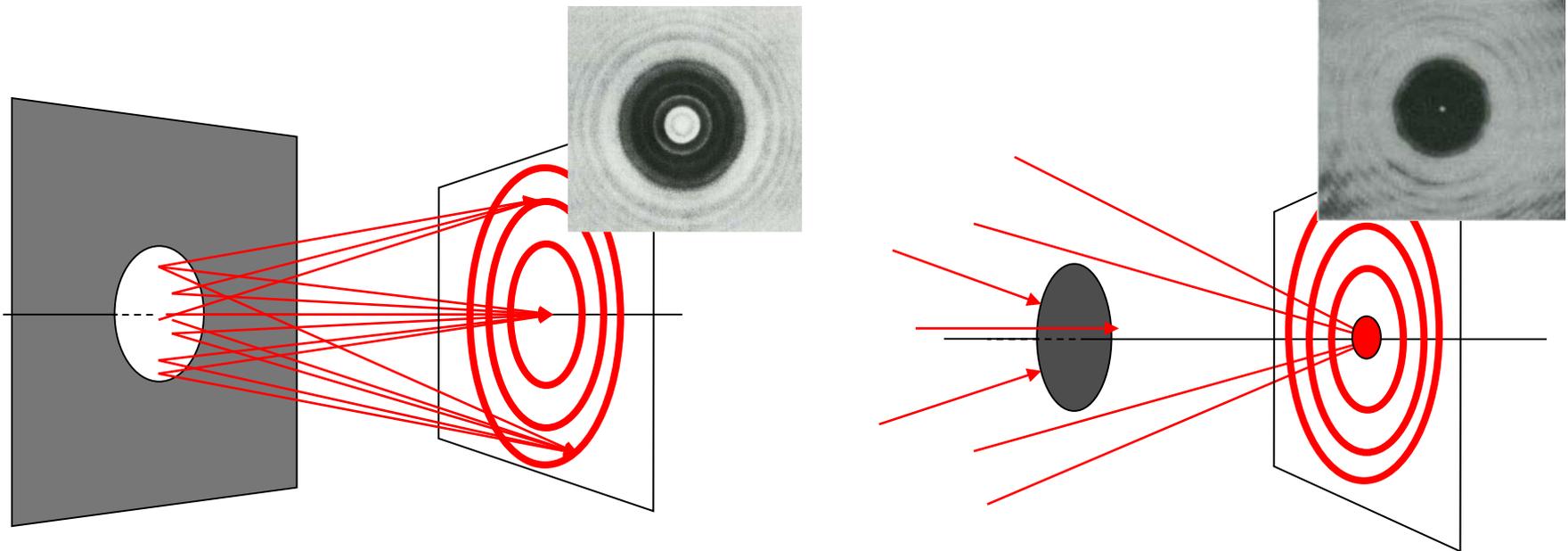
源屏距离： $R \sim 1\text{m}$

屏屏距离： $b \sim 3-5\text{m}$



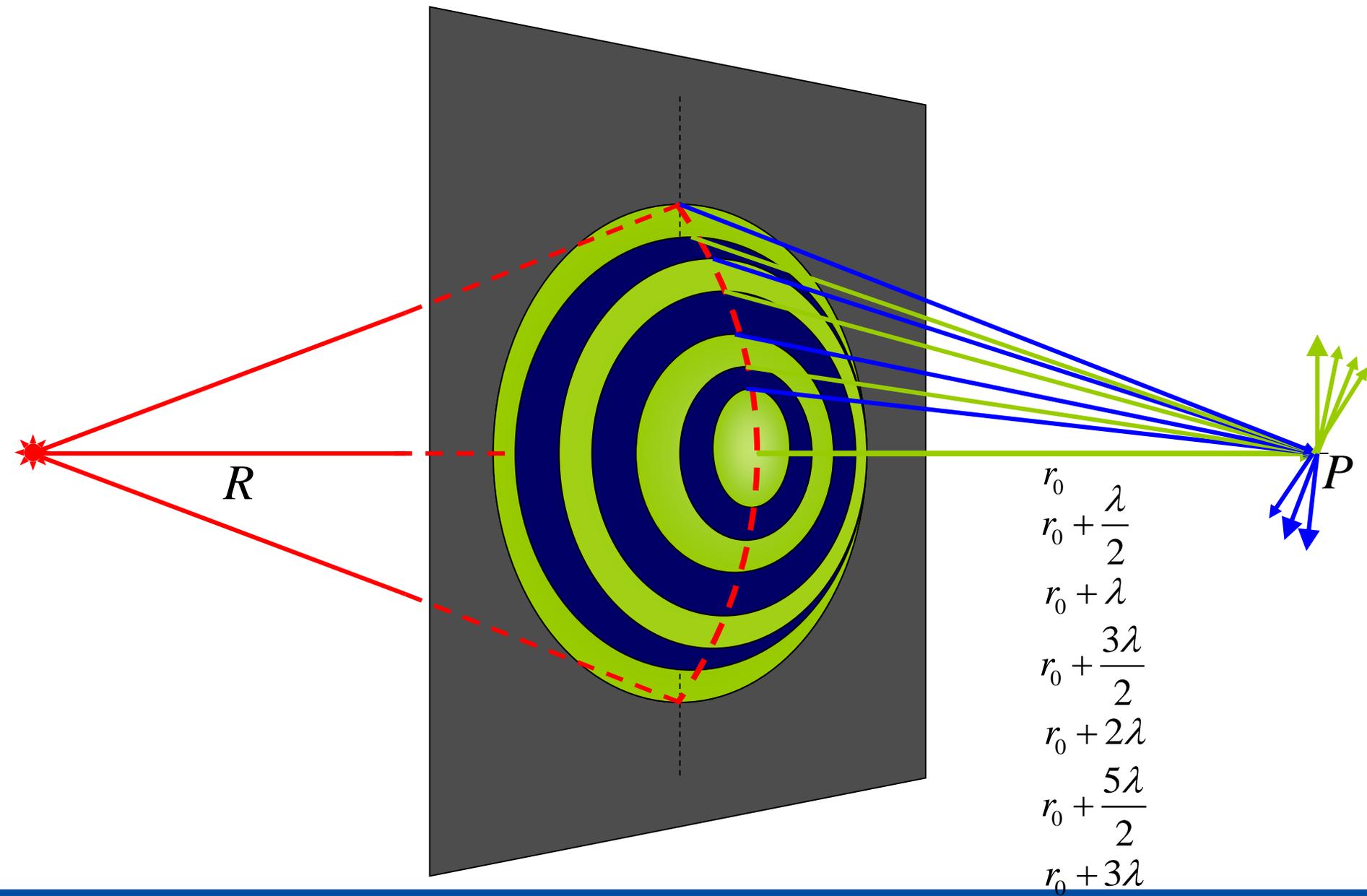
圆孔衍射：接收屏上可见同心圆环，接收屏沿轴向移动，圆环中心明暗交替变化。

圆屏衍射：接收屏上可见同心圆环，接收屏沿轴向移动，圆环中心永远是亮点。

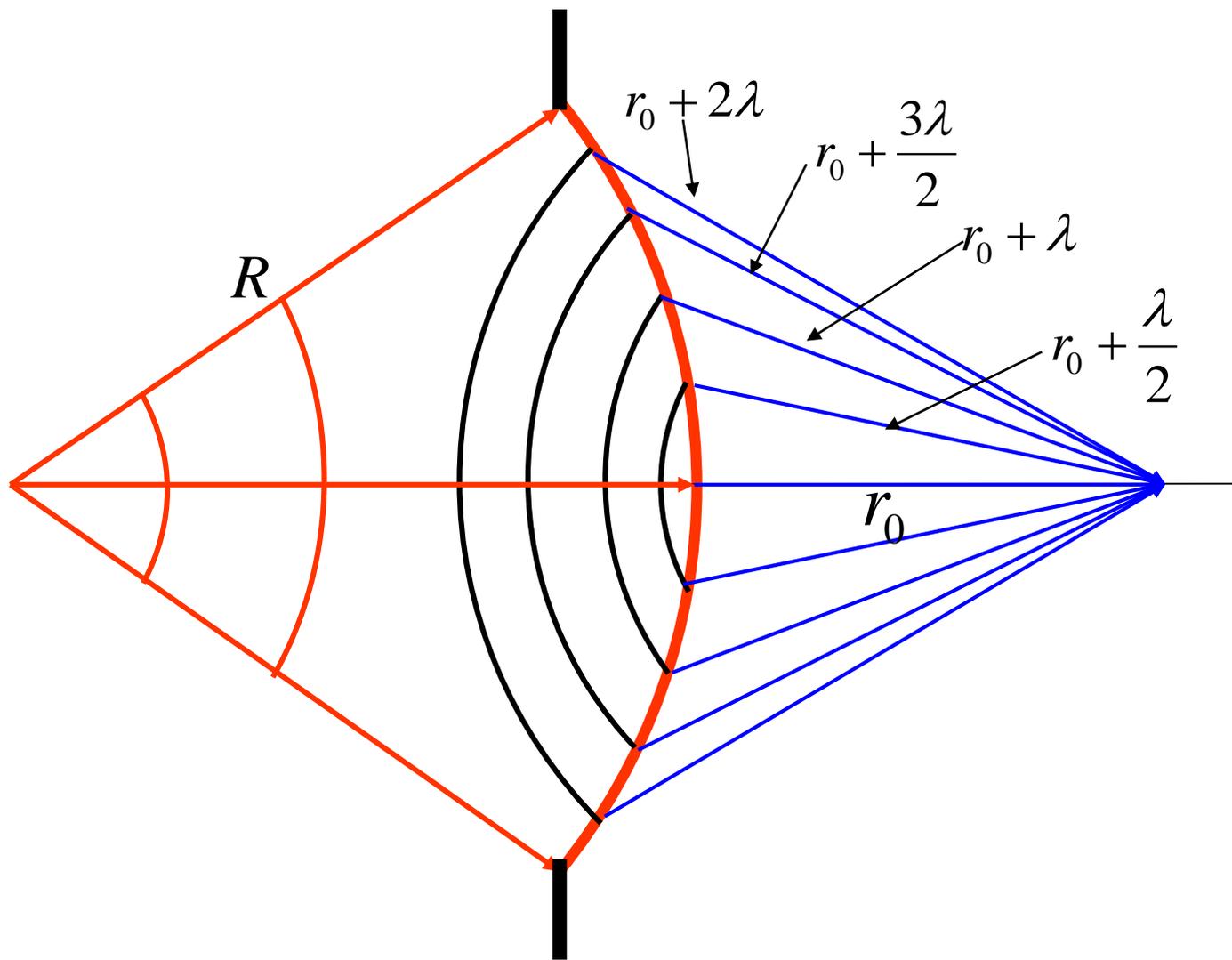


2. 半波带(half-wave zone)法

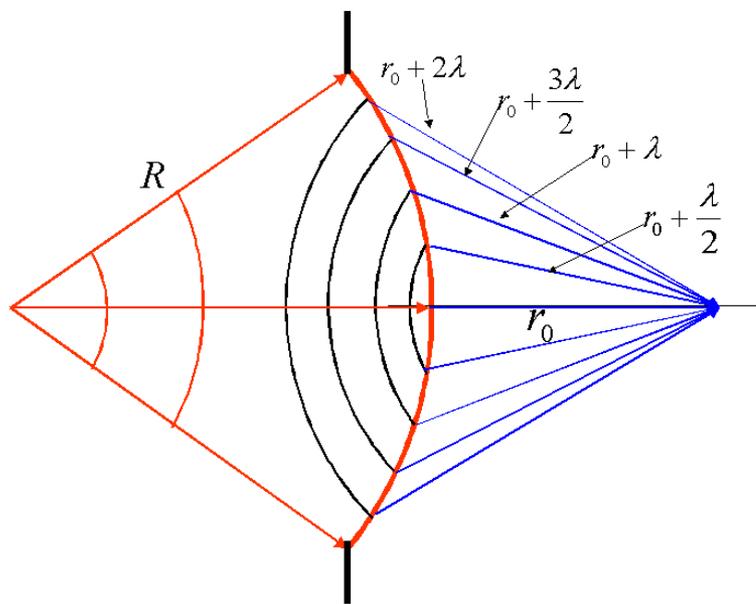
- 设法求解菲涅耳—基尔霍夫衍射积分公式。
- 将积分近似化为求和。
- 将波前（球面）划分为一系列的同心圆环带，每一带的中心到P点的距离依次相差半个波长。这些圆环带称为**半波带**。



- $r_0 + \frac{\lambda}{2}$
- $r_0 + \lambda$
- $r_0 + \frac{3\lambda}{2}$
- $r_0 + 2\lambda$
- $r_0 + \frac{5\lambda}{2}$
- $r_0 + 3\lambda$



半波带的次波



- 在球面上，各次波波源初位相相等。相邻半波带发出的次波，到达P点时，光程差为 $\lambda/2$ ，相位差为 π ，相位相反，振动方向相反，且振幅依次减小
- 相互抵消

- 通过近似求解积分公式：划分半波带

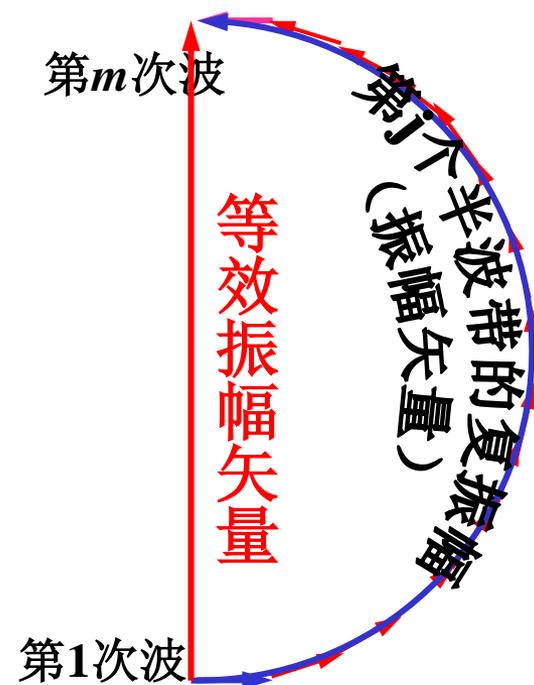
$$\tilde{U}(x, y) = \sum_{j=1}^N \frac{e^{-i\pi/2}}{\lambda} \iint_{\Sigma_j} \tilde{U}_0(x', y') \frac{\cos \theta_0 + \cos \theta}{2} \frac{e^{ikr}}{r} dx' dy' = \sum_{j=1}^N \tilde{U}_j$$

第j个半波带片

1. 求出每个半波带在场点的复振幅:

振幅矢量法：

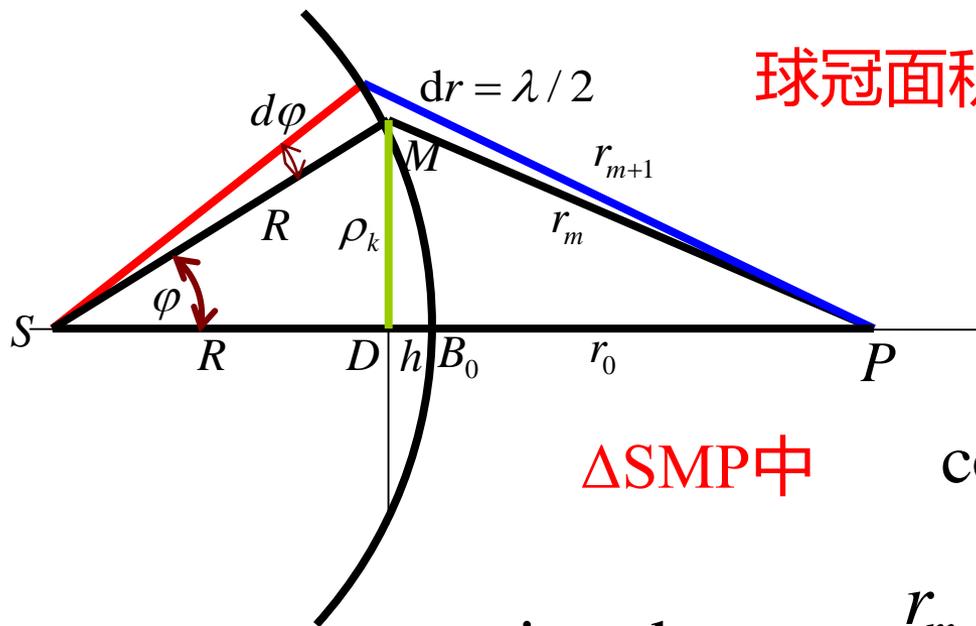
- i. 将每一半波带m等分，相邻次波相位差 π/m ;
- ii. 满足傍轴条件，各次波振幅近似相等



2. 下一步再处理各个半波带复振幅的相干叠加

计算各个半波带的面积 S_m

$$\tilde{U}(P) = K \oiint_{\Sigma} \tilde{U}(Q) F(\theta_0, \theta) \frac{e^{ikr}}{r} d\Sigma = K \sum_{m=1}^n \tilde{U}(Q) F(\theta_m) e^{ikr_m} \frac{S_m}{r_m}$$



球冠面积

$$S = 2\pi R h$$

$$= 2\pi R^2 (1 - \cos \varphi)$$

$$dS = 2\pi R^2 \sin \varphi d\varphi$$

ΔSMP 中

$$\cos \varphi = \frac{R^2 + (R + r_0)^2 - r_m^2}{2R(R + r_0)}$$

$$\sin \varphi d\varphi = \frac{r_m}{R(R + r_0)} dr_m \quad dS = \frac{2\pi R r_m}{R + r_0} dr_m$$

第m个半波带面积

$$\frac{S_m}{r_m} = \frac{\pi R}{R + r_0} \lambda$$

$$dS = S_m$$

$$dr_m = \lambda / 2$$

菲涅耳-基尔霍夫衍射积分公式

$$\begin{aligned}
 \tilde{U}(P) &= K \oiint_{\Sigma} \tilde{U}(Q) F(\theta_0, \theta) \frac{e^{ikr}}{r} d\Sigma \\
 &= K \sum_{m=1}^n \tilde{U}(Q) F(\theta_m) e^{ikr_m} \frac{S_m}{r_m} \\
 &= K \tilde{U} \frac{S_m}{r_m} \sum F(\theta_m) e^{i[\phi_0 + (m-1)\pi]} \\
 &= K \tilde{U} e^{i\phi_0} \frac{S_m}{r_m} \sum \frac{1}{2} (1 + \cos \theta_m) e^{i(m-1)\pi} \\
 &= \tilde{A} \sum_{m=1}^n (-1)^{m-1} (1 + \cos \theta_m) \quad \left(\tilde{A} = \frac{K \tilde{U} e^{i\phi_0}}{2} \frac{S_m}{r_m} \right)
 \end{aligned}
 \quad \frac{S_m}{r_m} = \frac{\pi R}{R + r_0} \lambda$$

$$\tilde{U}(P) = \tilde{A} \sum_{m=1}^n (-1)^{m-1} (1 + \cos \theta_m) = \sum_{m=1}^n \tilde{U}_m = \sum_{m=1}^n A_m (-1)^{m-1}$$

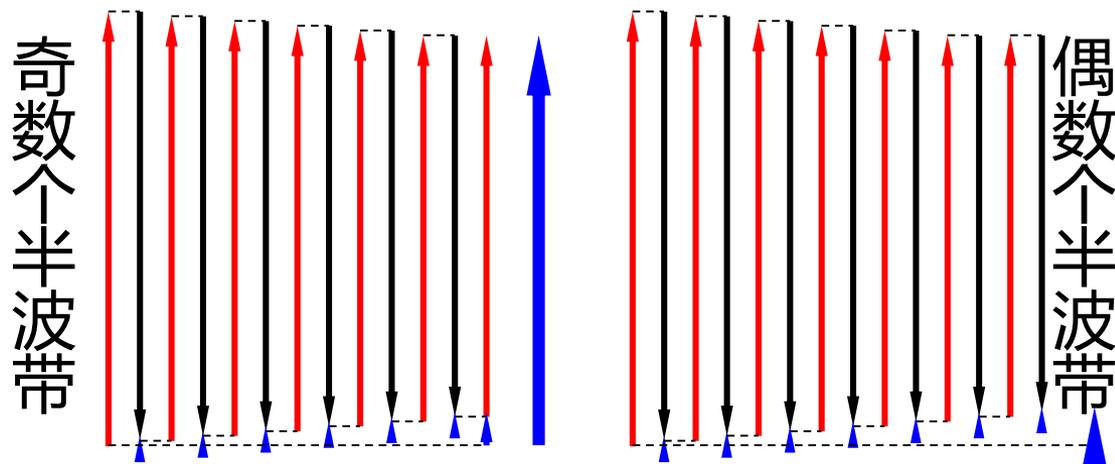
$$\tilde{U}_m = \tilde{A} (1 + \cos \theta_m) (-1)^{m-1}$$

第 m 个半波带发出的次波在 P 点的复振幅

$$A_m = |\tilde{A}| (1 + \cos \theta_m)$$

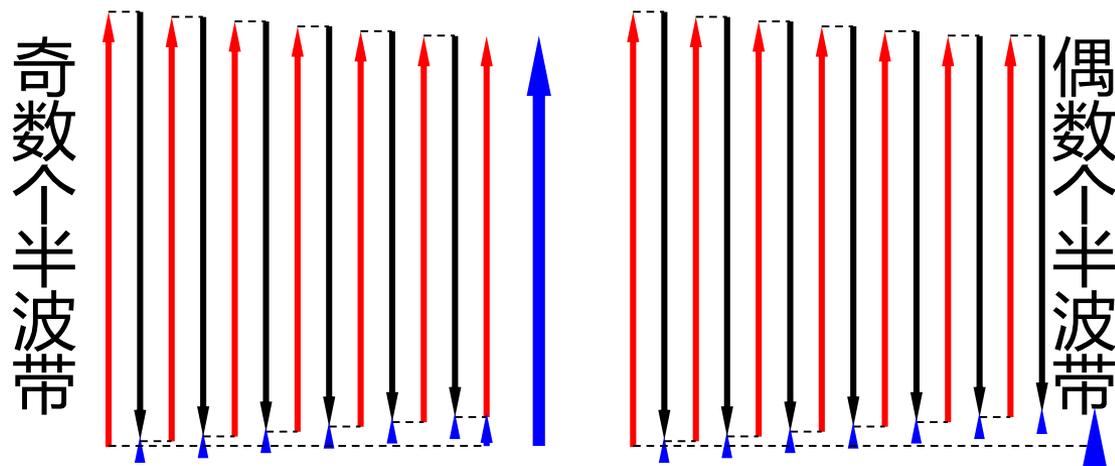
取孔中心次波相位为 0, A_m 为第 m 个半波带发出的次波在 P 点的振幅

P 点处：相邻波带次波的位相相反； m 越大的波带，振幅越小。



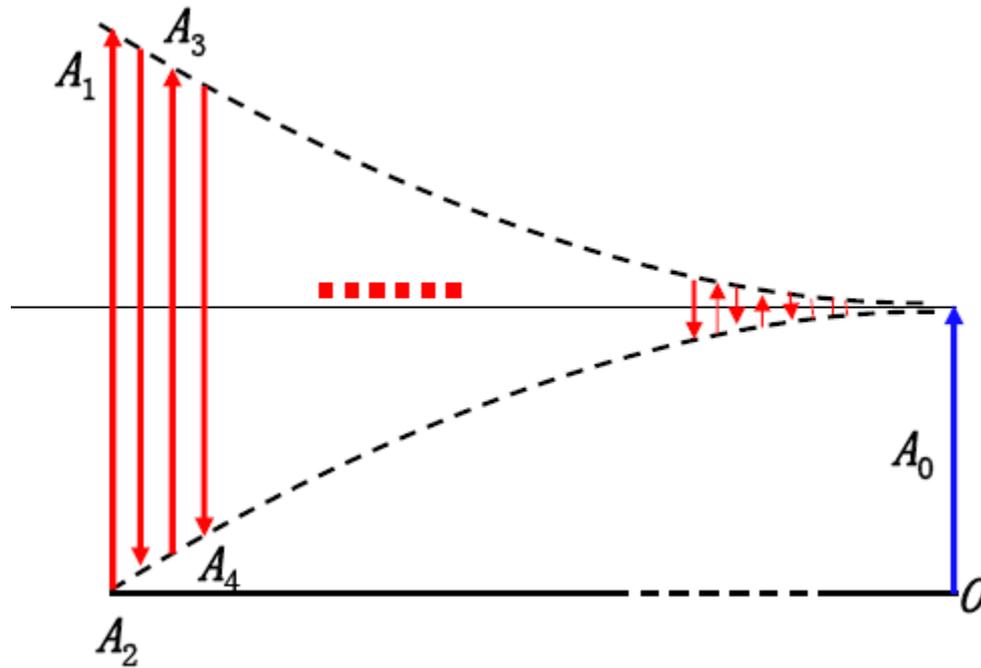
$$\begin{aligned}\tilde{U}(P) &= \sum_{m=1}^n (-1)^{m-1} A_m = \frac{1}{2} A_1 + \left(\frac{1}{2} A_1 - A_2 + \frac{1}{2} A_3\right) + \left(\frac{1}{2} A_3 - A_4 + \frac{1}{2} A_5\right) + \dots \\ &= \frac{1}{2} [A_1 + (-1)^{n-1} A_n]\end{aligned}$$

解释：波带数n为奇数，亮点；n为偶数，暗点



$$\tilde{U}(p) = \frac{1}{2} [A_1 + (-1)^{n-1} A_n]$$

自由传播 $n \rightarrow \infty$ $A_n \rightarrow 0$ $A(P) = \frac{1}{2} A_1$ 始终亮点



圆屏，前n个半波带被遮住

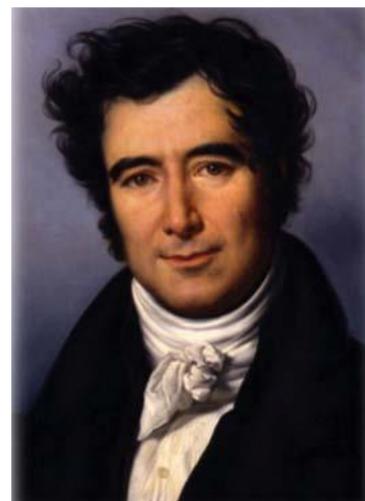
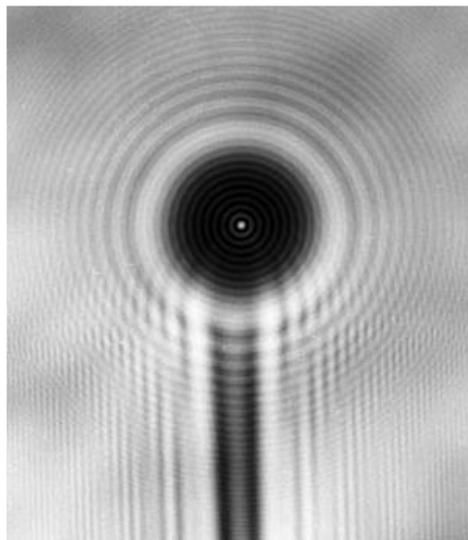
$$A(P) = \sum_{n+1}^{\infty} A_k = \frac{1}{2} A_{n+1}$$

总是亮点:泊松亮斑 (阿拉果亮斑)

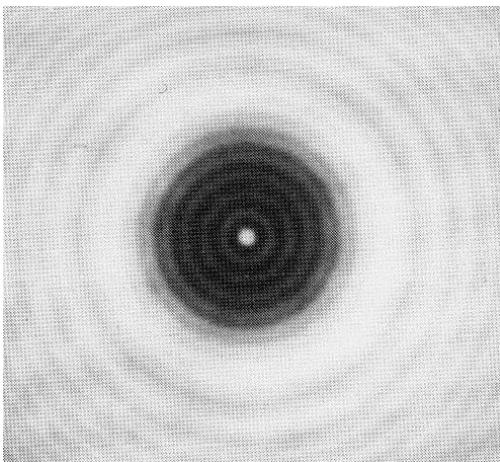
惠更斯-费涅耳原理**1818**巴黎科学院的一次科学竞赛中提出，倾向微粒说的评委泊松据此算出圆屏衍射的中心竟会是一亮斑（泊松亮斑），觉得十分荒谬，影子中间怎么会出亮点呢？但随后阿拉果的实验观察到了圆屏衍射的中心亮斑，位置亮度和理论符合得相当完美。



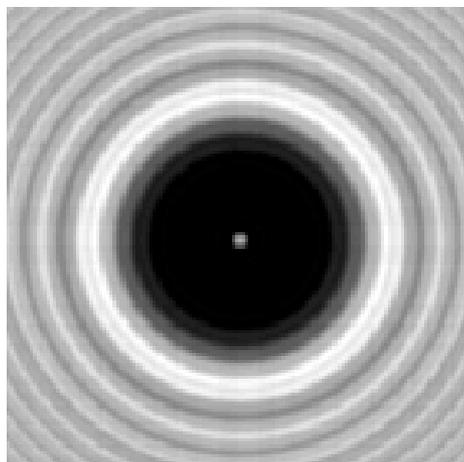
Siméon Poisson



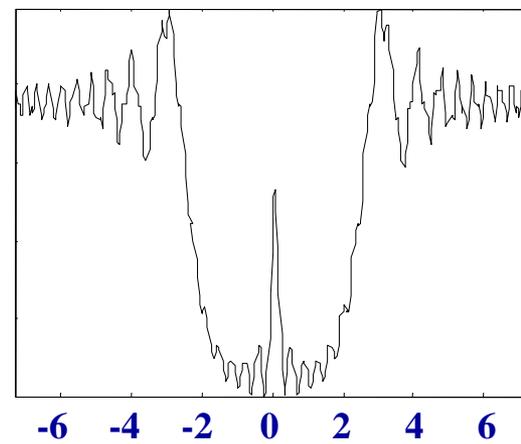
Francois Arago



(a) 实验图样



(b) 仿真图样

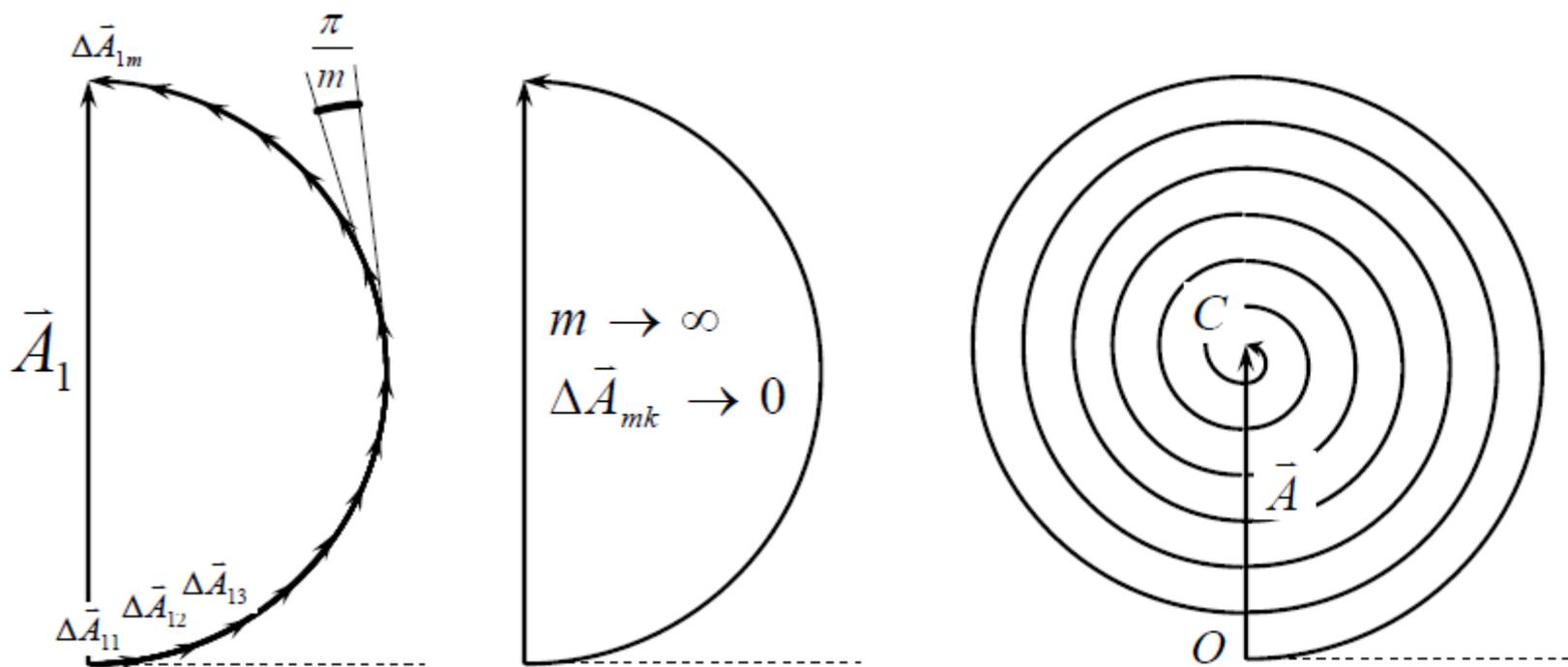


(c) 相对强度分布

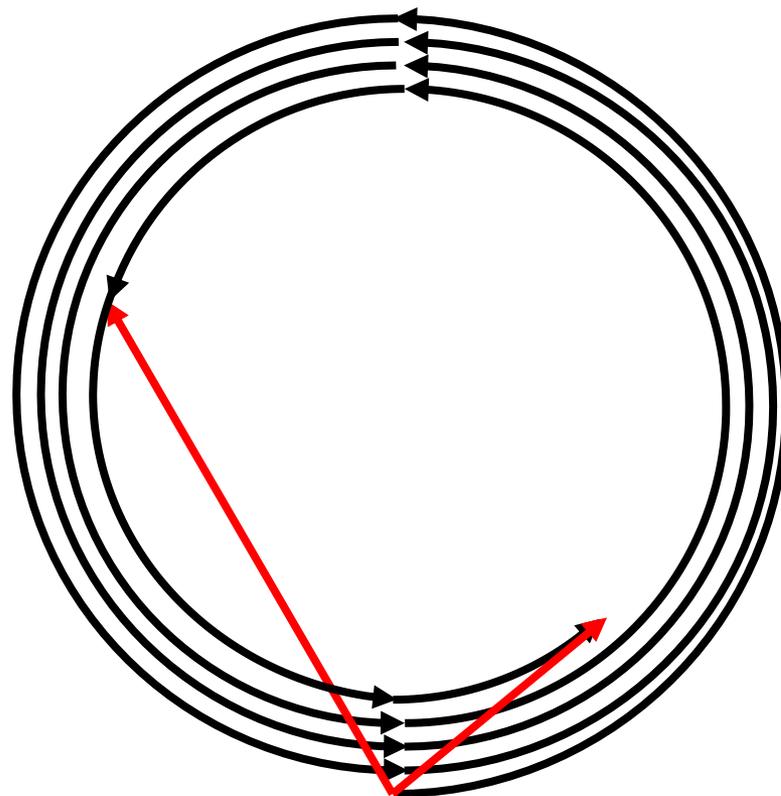
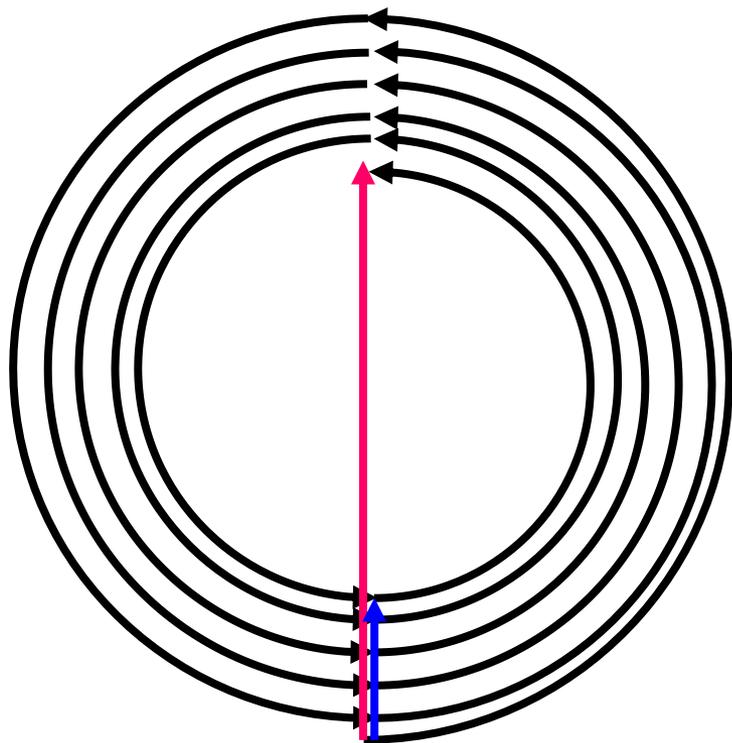
圆盘的菲涅耳衍射

3. 矢量图解(vector diagram)法

- 可以将任何一个半波带进一步细分为 n 个，得到更多的波带，相邻波带间光程差为 $\lambda/2n$ ，位相差为 $\frac{\pi}{n}$ 。 n 很大时，位相差很小，用振幅矢量法，原来的每个半波带的波矢变为由 n 个小波矢组成的半圆。



半波带的弧形振幅矢量和非整数个半波带

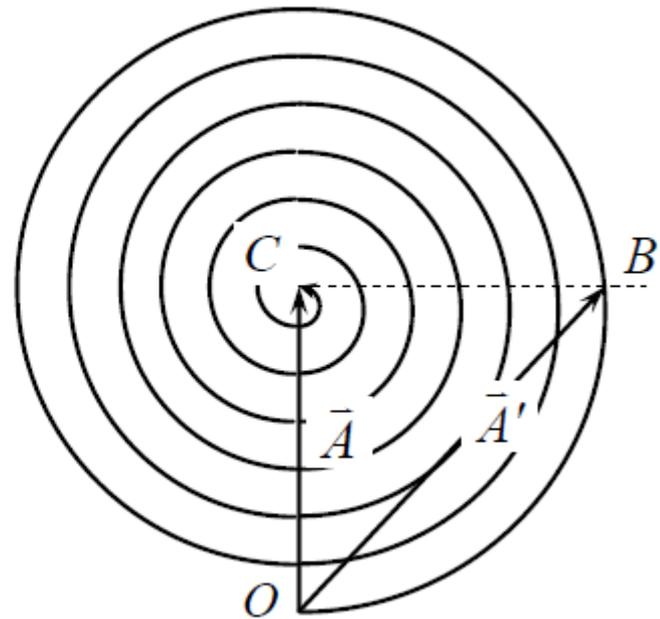


例：圆孔包含**1/2**个半波带时轴上的衍射强度

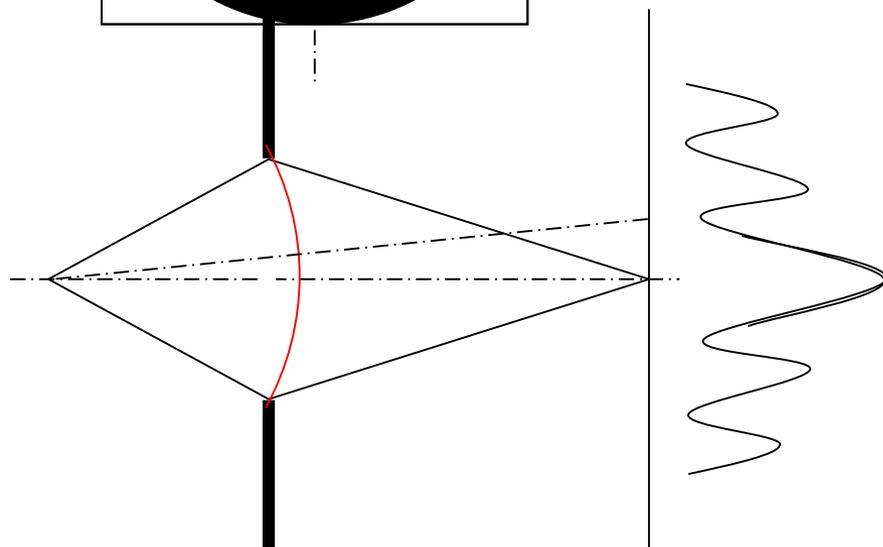
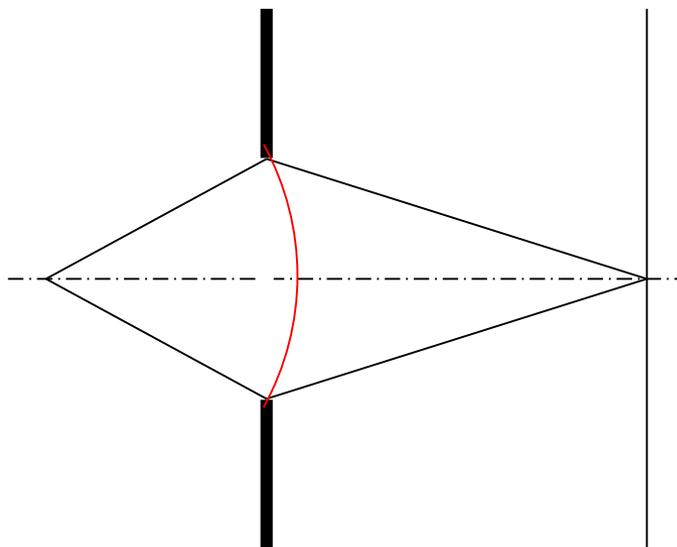
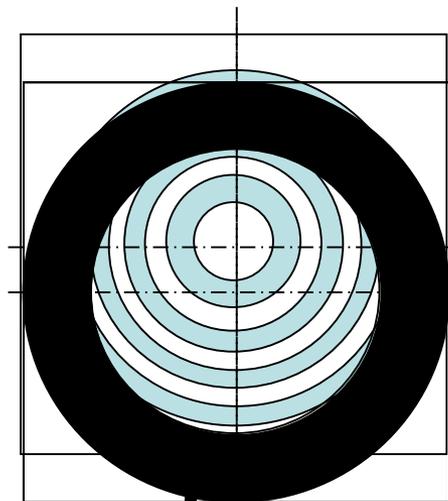
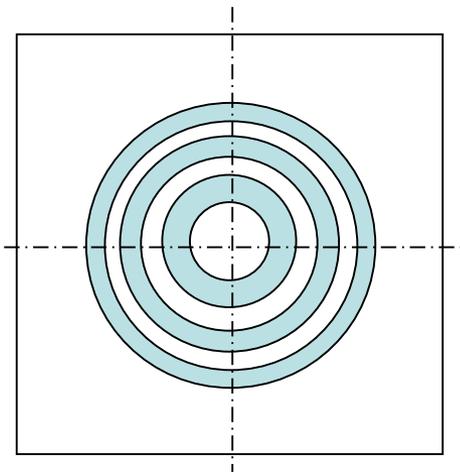
边缘与中心光程差 $\lambda/4$ ，位
相差 $\pi/2$ ，振动曲线为**OB**

$$A' = OB = \sqrt{2}A$$

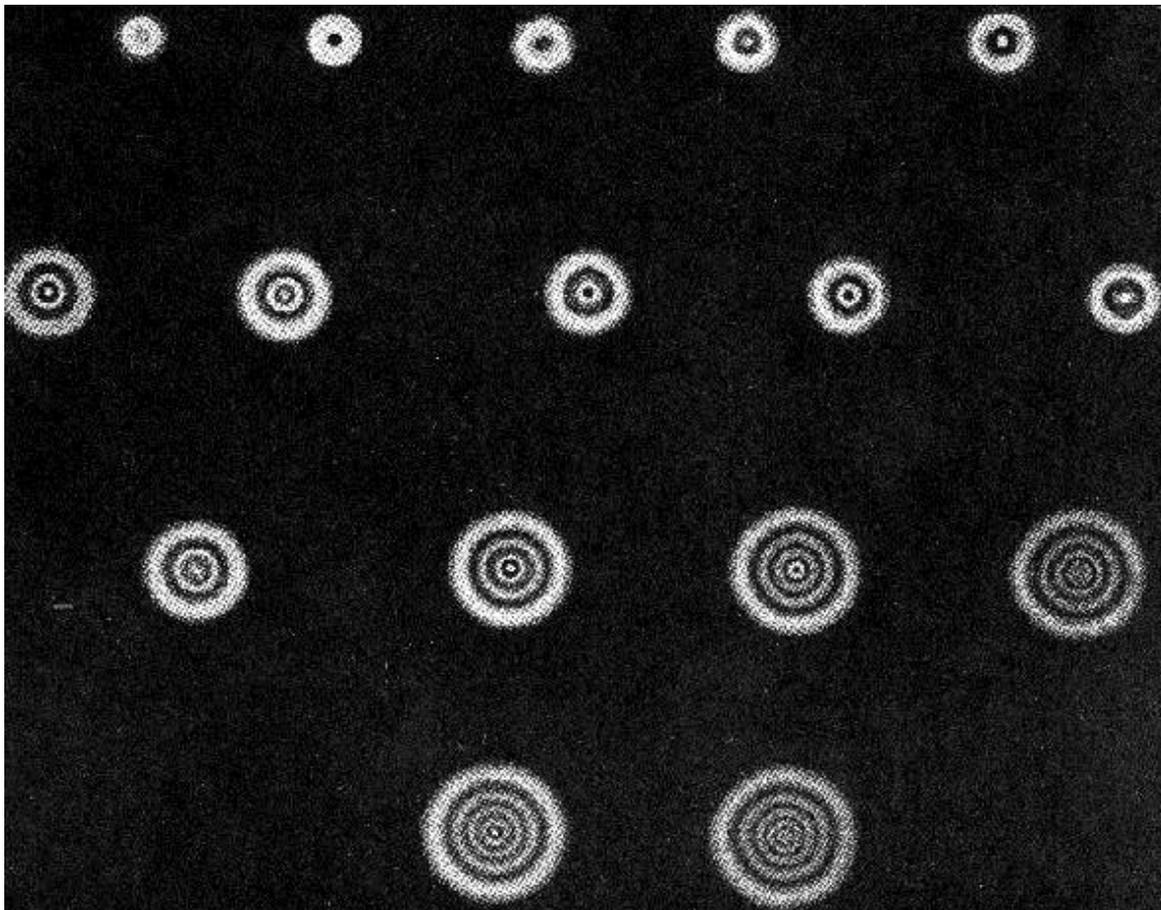
$$I' = 2A^2$$



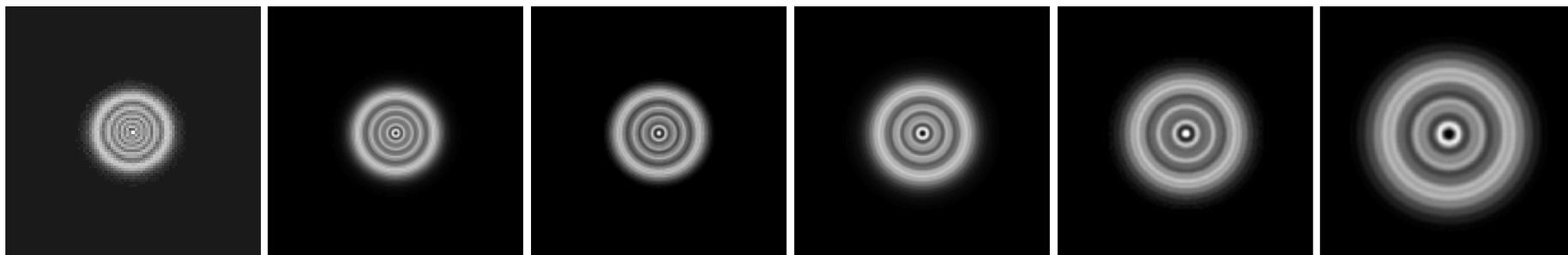
菲涅耳圆孔衍射花样



圆孔的菲涅耳衍射图样



距离衍射屏不同观察平面上的衍射图样



(a) $z=1.14\text{m}$

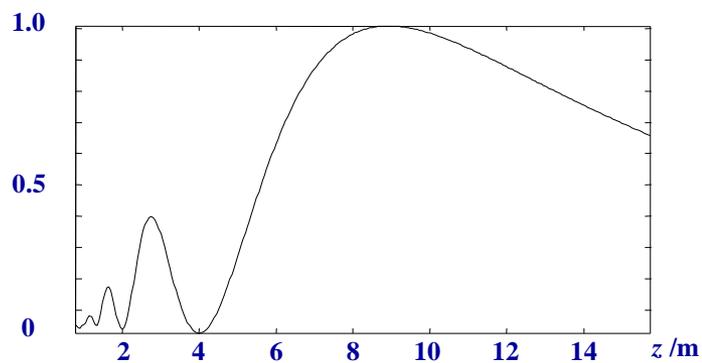
(b) $z=1.35\text{m}$

(c) $z=1.60\text{m}$

(d) $z=2.00\text{m}$

(e) $z=2.70\text{m}$

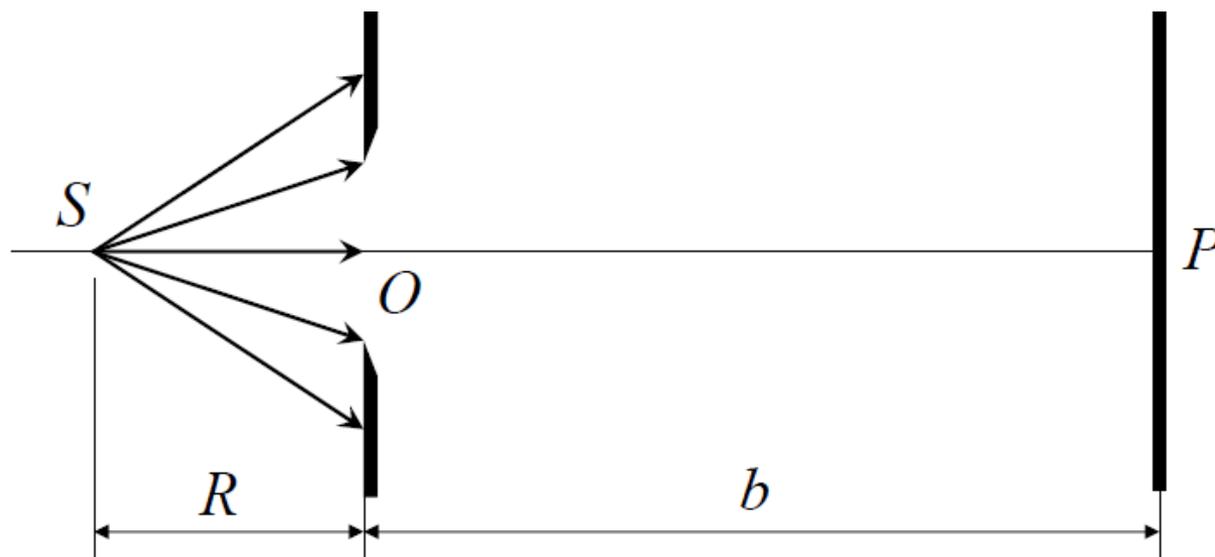
(f) $z=4.00\text{m}$



(g) 衍射图样中心的相对强度

圆孔的菲涅耳衍射仿真图样（不同观察平面上， z : 观察平面到衍射屏平面的距离）

例：自由传播验证比例系数 $K = -i/\lambda$



自由传播的球面波：

$$\tilde{U} = \frac{a}{r} e^{ikr}$$



在衍射屏处： $\tilde{U}(O) = \frac{a}{R} e^{ikR}$

在屏幕处： $\tilde{U}(P) = \frac{a}{R+b} e^{ik(R+b)}$

$$\tilde{U}(P) = \frac{a}{R+b} e^{ik(R+b)} = \frac{1}{2} \tilde{U}_1(P)$$

$$\tilde{U}_1(P) = K \iint_{\text{第一个半波带}} \tilde{U}_0(O) F(\theta_0, \theta) \frac{e^{ikr}}{r} d\Sigma$$

对第一个半波带: $F(\theta_0, \theta) \cong 1$

$$\begin{aligned} \tilde{U}_1(P) &= K \int_b^{b+\lambda/2} \tilde{U}_0(O) e^{ikr} dr \\ &= \frac{2\pi a K}{R+b} e^{ikR} \int_b^{b+\lambda/2} e^{ikr} dr \\ &= \frac{2\pi a K}{ik(R+b)} e^{ik(R+b)} (e^{ik\lambda/2} - 1) \\ &= -\frac{4\pi a K}{ik(R+b)} e^{ik(R+b)} \\ &= -\frac{2\lambda a K}{i(R+b)} e^{ik(R+b)} \end{aligned}$$

$$\tilde{U}_0(O) = \tilde{U}(O) = \frac{a}{R} e^{ikR}$$

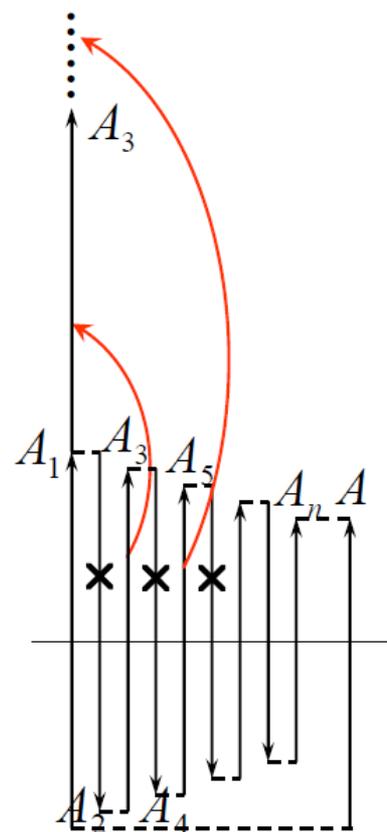
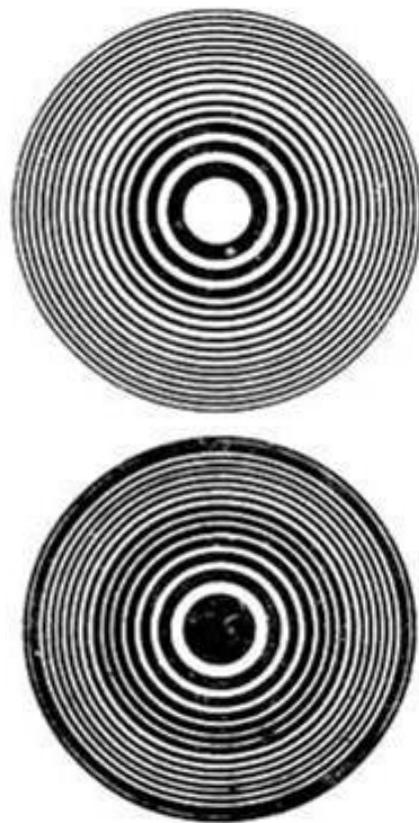
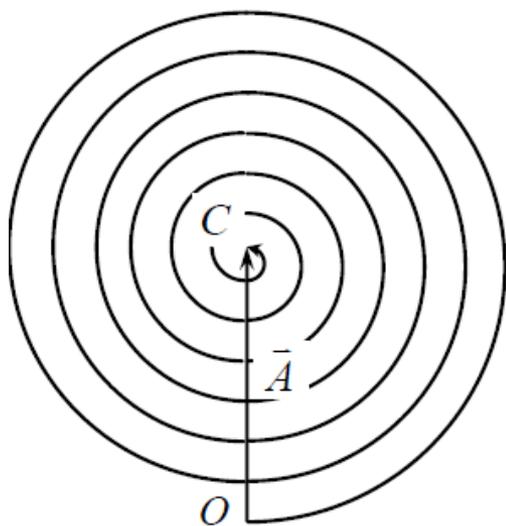
$$\frac{d\Sigma}{r} = \frac{2\pi R}{R+b} dr$$

与自由传播到P点的比较:

$$-\frac{\lambda K}{i} = 1 \quad \text{或} \quad K = -\frac{i}{\lambda}$$

4. 费涅耳波带片(zone plate)

- 将偶数或奇数半波带涂黑或转相，即构成了费涅耳波带片。
- 透过波带片的光，在场点P处光程差依次为 λ ，相位相同，振动方向也相同，合振动大大增强，衍射后的光强大大增强。
- 相当于将光波汇聚到P点。

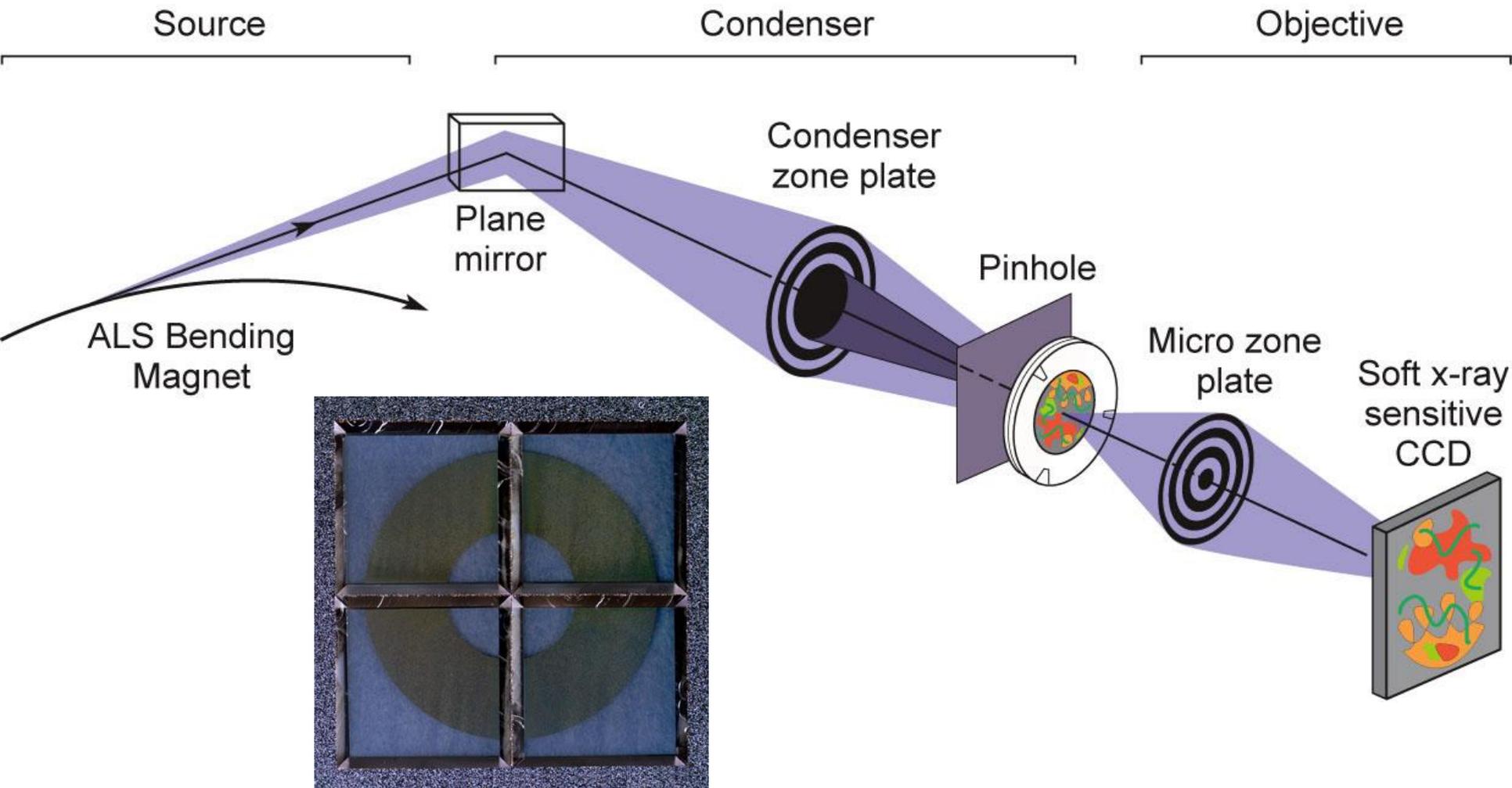


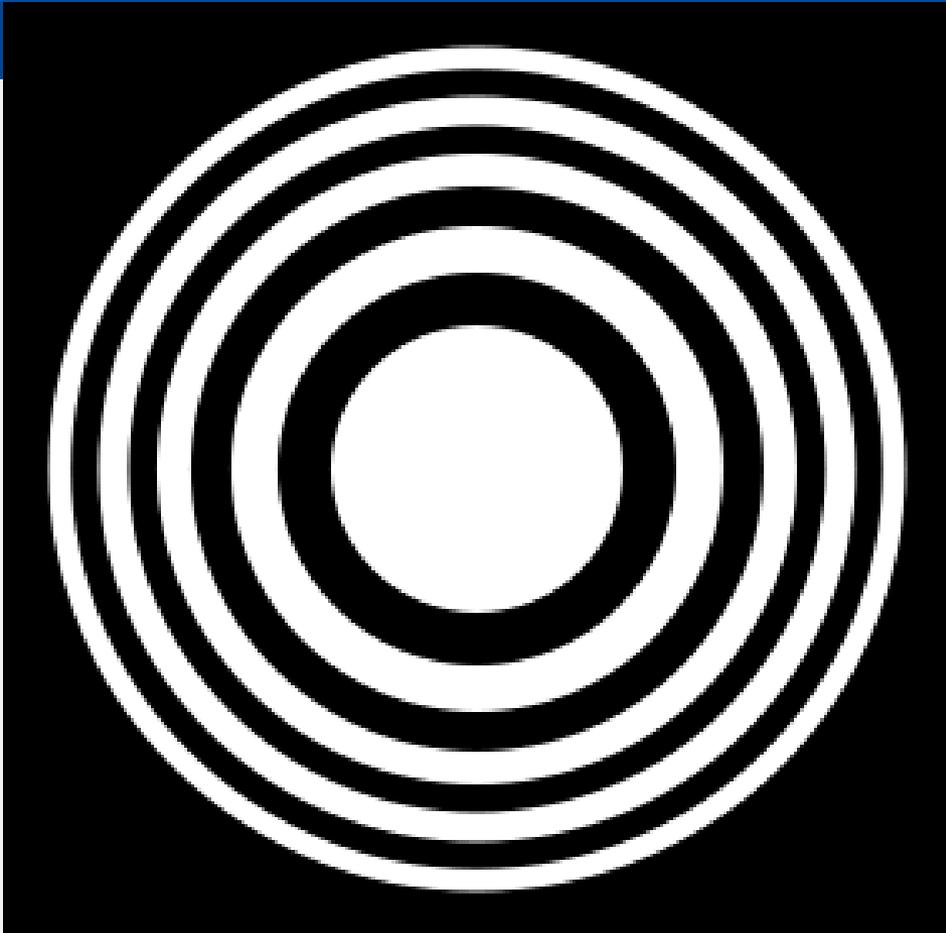
例：20个半波带，1、3、5、.....、19共10个奇数半波带透光，2、4、6、.....、20共10个偶数半波带涂黑，场点的光强

$$A' = A_1 + A_2 + \dots + A_{19} \approx 10A_1 = 20A$$

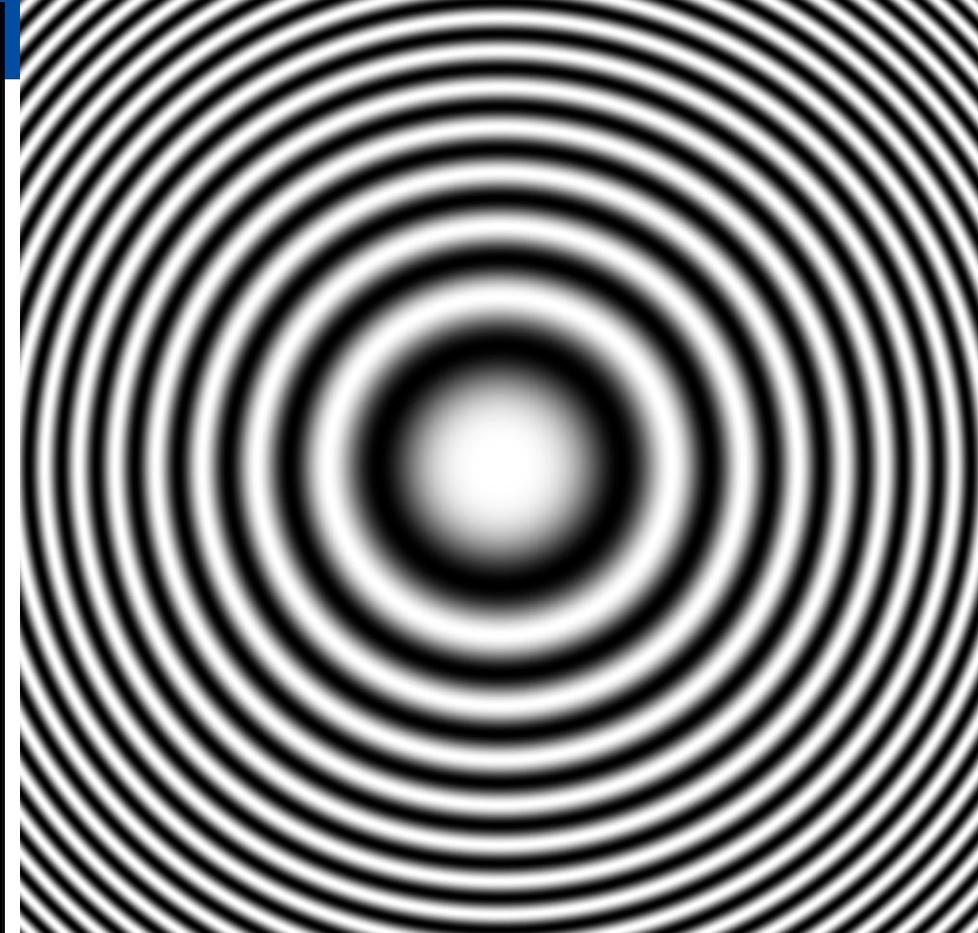
$$I' = (A')^2 = 400A^2$$

用于同步辐射软x射线的波带片





黑白型



正弦型

半波带半径

$$r_n^2 - r_0^2 = \left(r_0 + n \frac{\lambda}{2}\right)^2 - r_0^2 = nr_0\lambda + \left(\frac{n}{2}\lambda\right)^2 \approx n\lambda r_0$$

$$\rho_n^2 = r_n^2 - (r_0 + h)^2 = r_n^2 - r_0^2 - 2r_0h - h^2$$

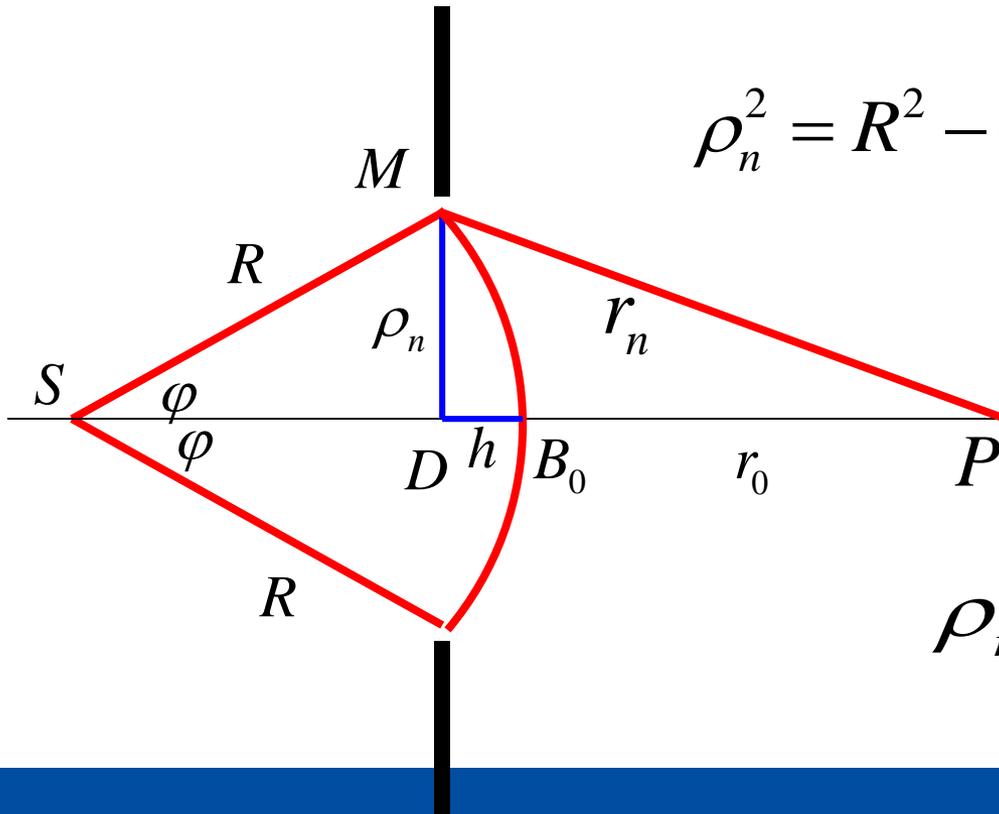
$$\approx n\lambda r_0 - 2r_0h$$

$$\rho_n^2 = R^2 - (R - h)^2 = 2Rh - h^2 \approx 2Rh$$

$$n\lambda r_0 - 2r_0h = 2Rh$$

$$h = \frac{nr_0}{2(R + r_0)} \lambda$$

$$\rho_n^2 \approx 2Rh = \frac{nr_0 R}{R + r_0} \lambda$$

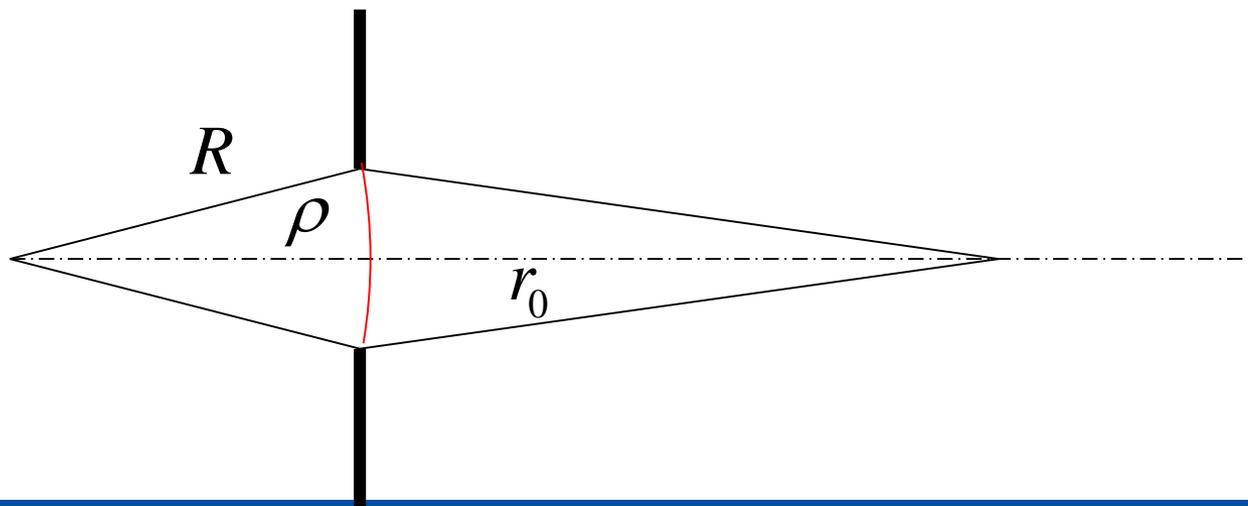


半波带片方程

$$\rho_n^2 = \frac{nr_0R}{R+r_0} \lambda \quad n = \frac{\rho_n^2}{\lambda} \frac{R+r_0}{Rr_0}$$

$$n = \frac{\rho^2}{\lambda} \left(\frac{1}{r_0} + \frac{1}{R} \right) \quad \text{半波带方程} \quad \frac{1}{r_0} + \frac{1}{R} = \frac{1}{\rho^2 / n\lambda}$$

n 的数值及奇偶性由 r_0 决定。



- 半波带方程 $\frac{1}{R} + \frac{1}{r_0} = \frac{n\lambda}{\rho^2} = \frac{1}{f} \quad f = \frac{\rho^2}{n\lambda}$

- 等效于透镜的Gauss公式

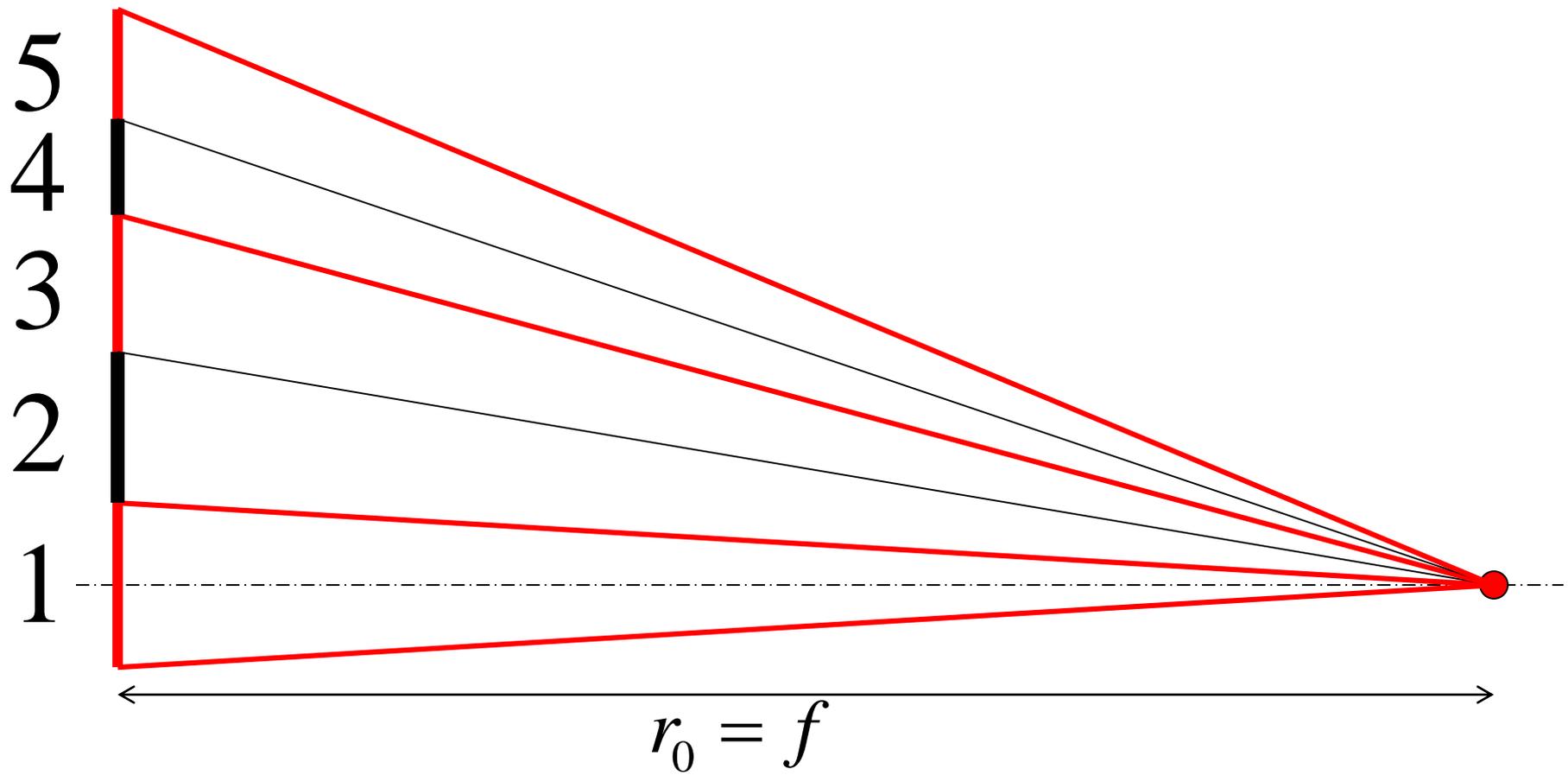
- 对于某一个波长，波带片的焦距是固定的。

- 对平行光，波带片为平面的。

- 在距离 r_0 处看来，半径为 ρ 的圆孔处是第 n 个半波带。

$$n = \frac{\rho^2}{\lambda} \frac{1}{r_0}$$

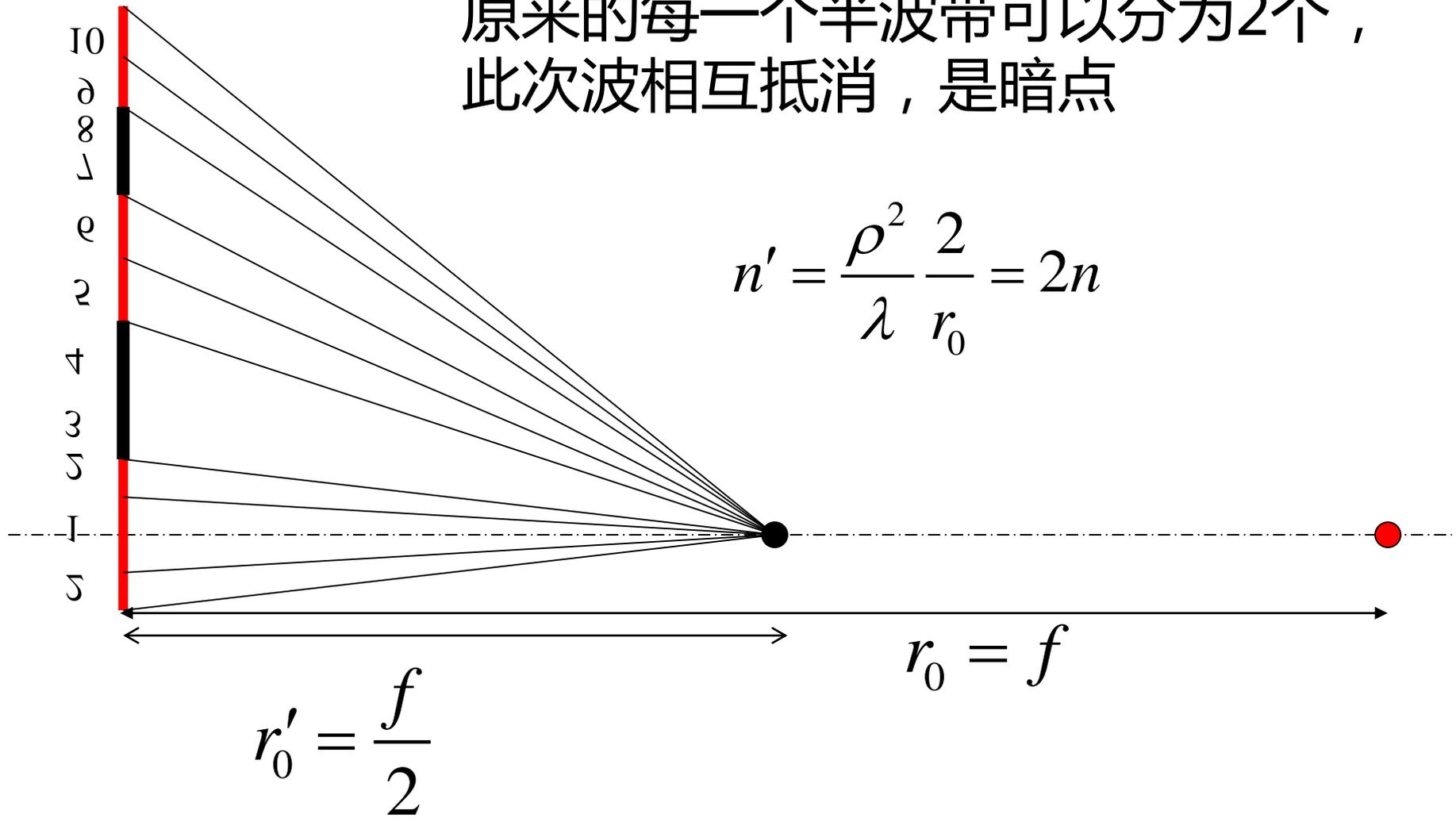
- 但除主焦点之外，还有许许多多个次焦点。



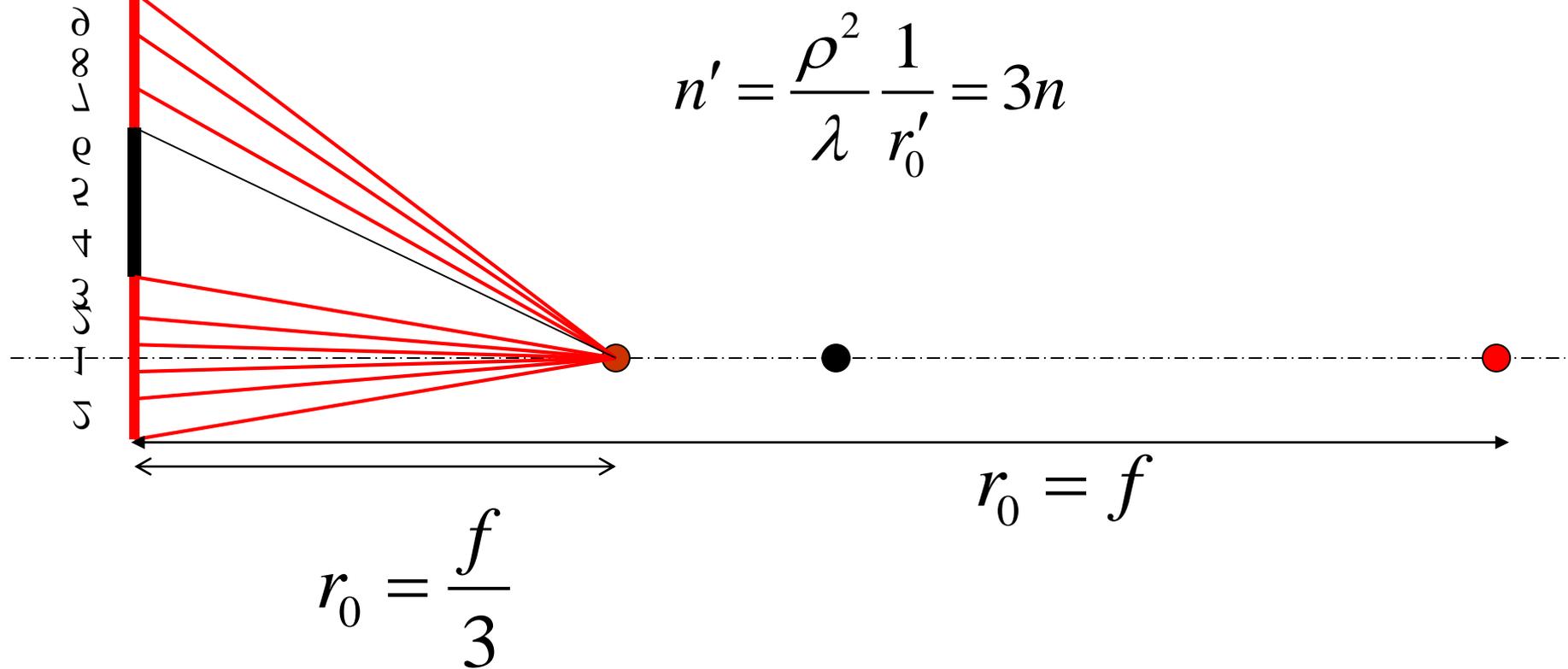
$$n = \frac{\rho^2}{\lambda} \left(\frac{1}{r_0} + \frac{1}{R} \right)$$

$$n = \frac{\rho^2}{\lambda} \frac{1}{r_0}$$

原来的每一个半波带可以分为2个，
此次波相互抵消，是暗点



原来的每一个半波带分为3个，其中2个的次波抵消，还剩余1个，为次亮点，即次焦点。



$$n = \frac{\rho^2}{\lambda} \left(\frac{1}{r_0} + \frac{1}{R} \right)$$

$$n = \frac{\rho^2}{\lambda} \frac{1}{r_0}$$

- 当波带片不变时， r_0 改变，会引起 n 的改变，即可划分的半波带数目改变。
- r_0 减小，到 $r_0/2$ 时， $n'=2n$ ，暗点；
- r_0 减小，到 $r_0/3$ 时， $n'=3n$ ，亮点，次焦点；
- r_0 减小，到 $r_0/4$ 时， $n'=4n$ ，暗点.....

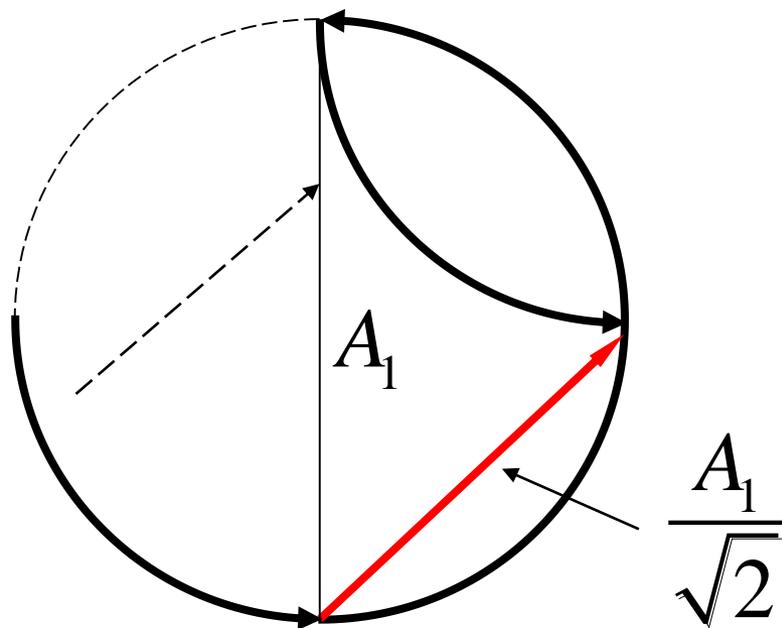
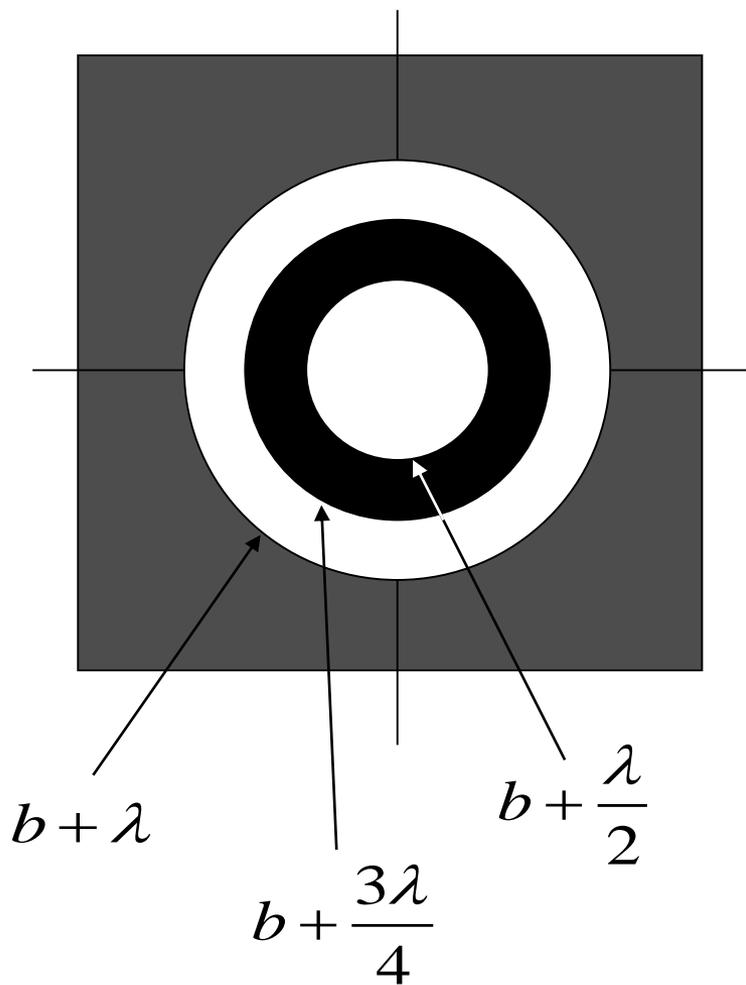
$$f' = \frac{f}{3}, \frac{f}{5}, \dots, \frac{f}{2m+1}$$

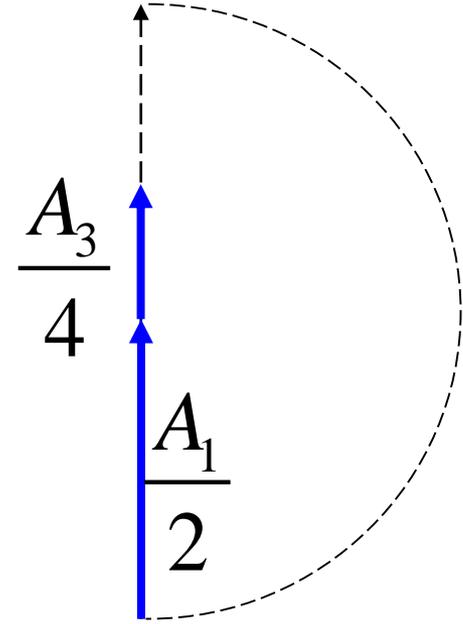
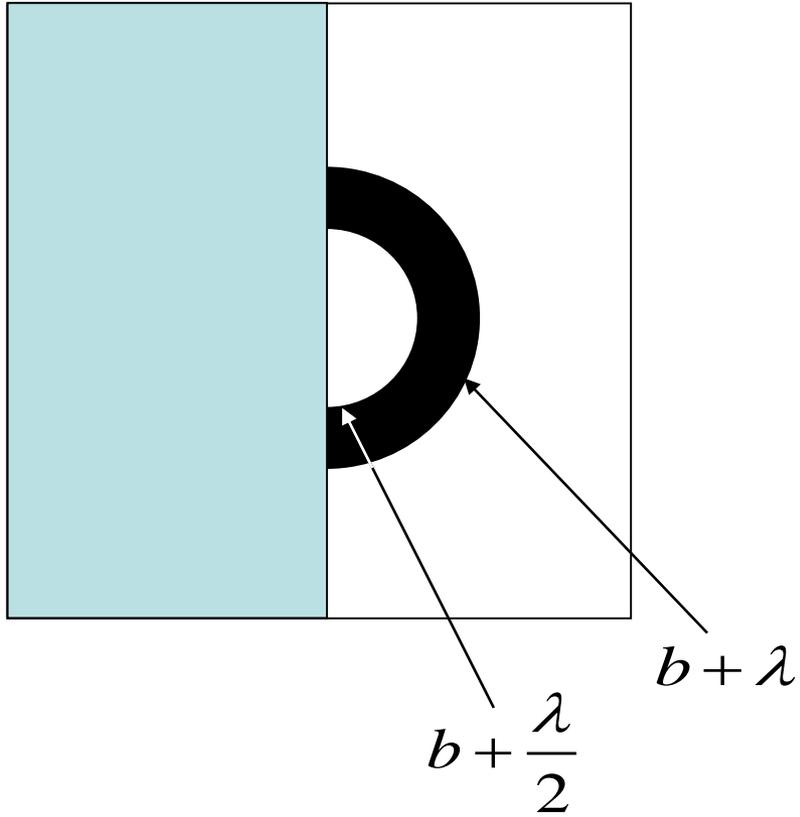
一系列次焦点

$$f' = -f, -\frac{f}{3}, -\frac{f}{5}, \dots, -\frac{f}{2m+1}$$

一系列虚焦点

矢量法求解菲涅耳衍射问题





$$\frac{A_1}{2} + \frac{A_3}{4} \approx \frac{3A_1}{4}$$

作业 : P.207-209: 1, 3, 5, 6, 8, 10

补充A：对透镜成像的再认识，Kinoform透镜

补充B：费涅耳矩孔衍射

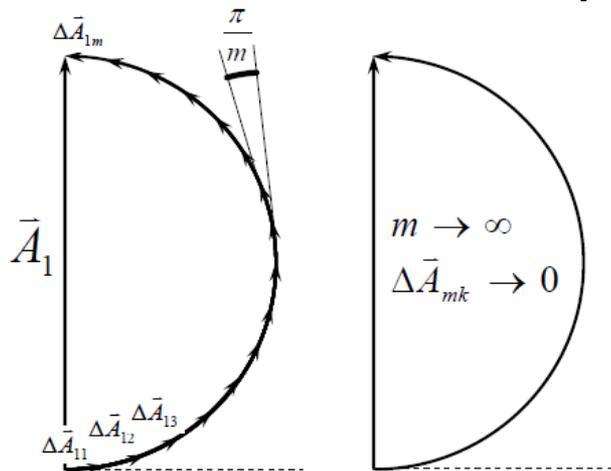
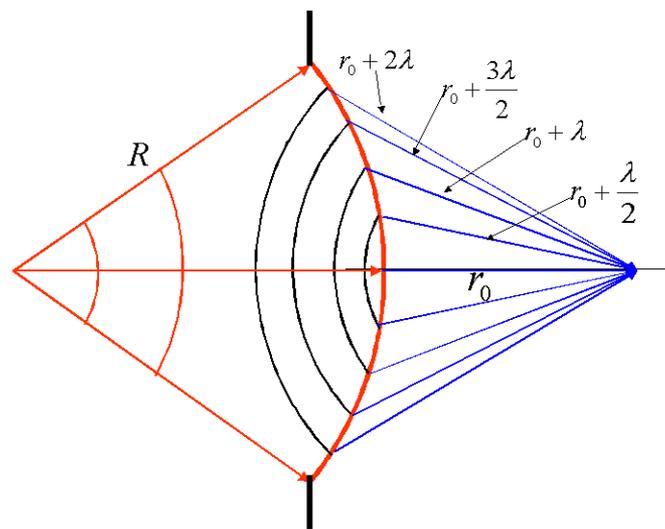
1 参数

2 傍轴条件下的光程

3 费涅耳积分和考纽螺线

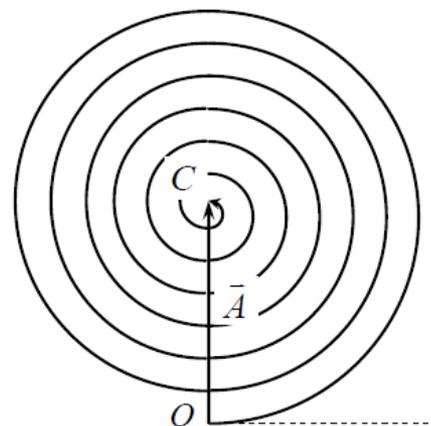
4 衍射强度

A : 对透镜成像的再认识

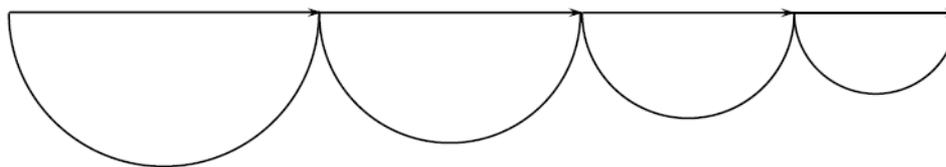
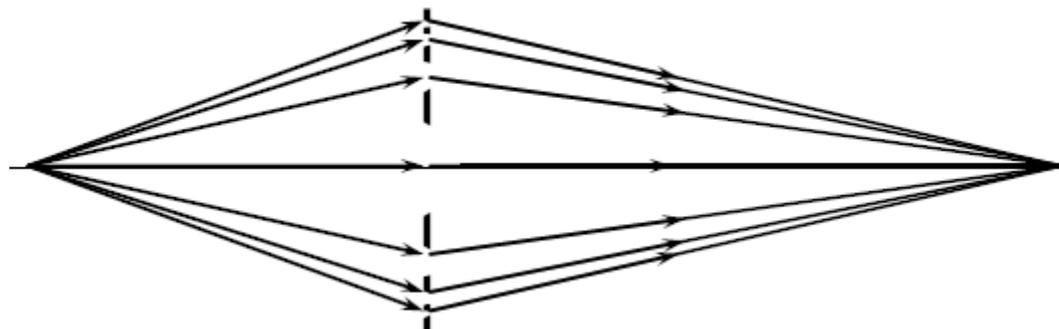
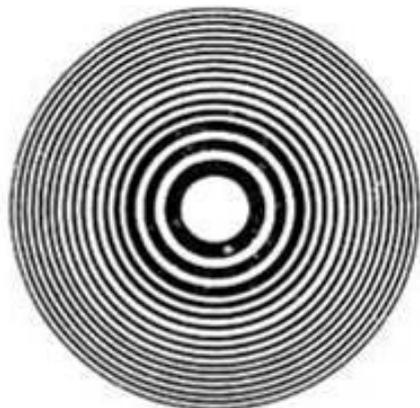


$$m \rightarrow \infty$$

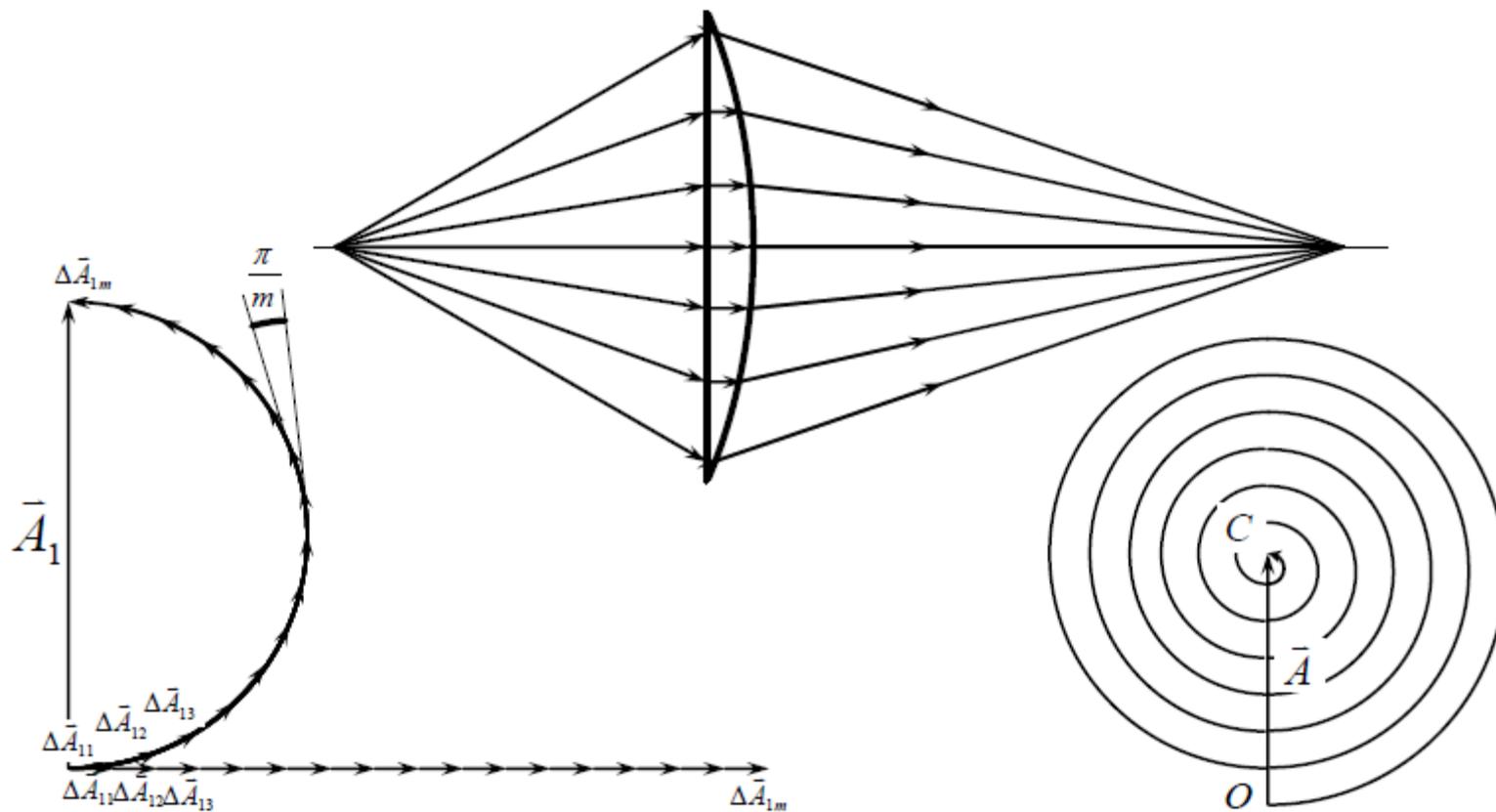
$$\Delta \bar{A}_{mk} \rightarrow 0$$



波带片



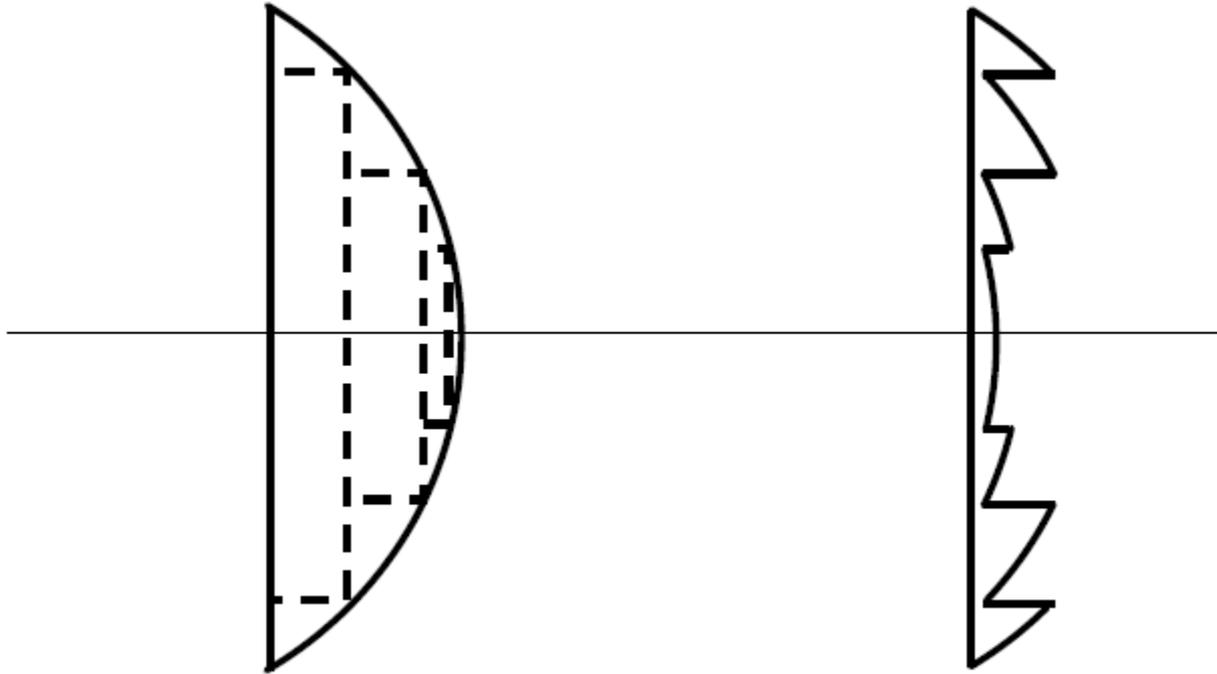
透镜



费马原理有着深刻的波动光学基础！

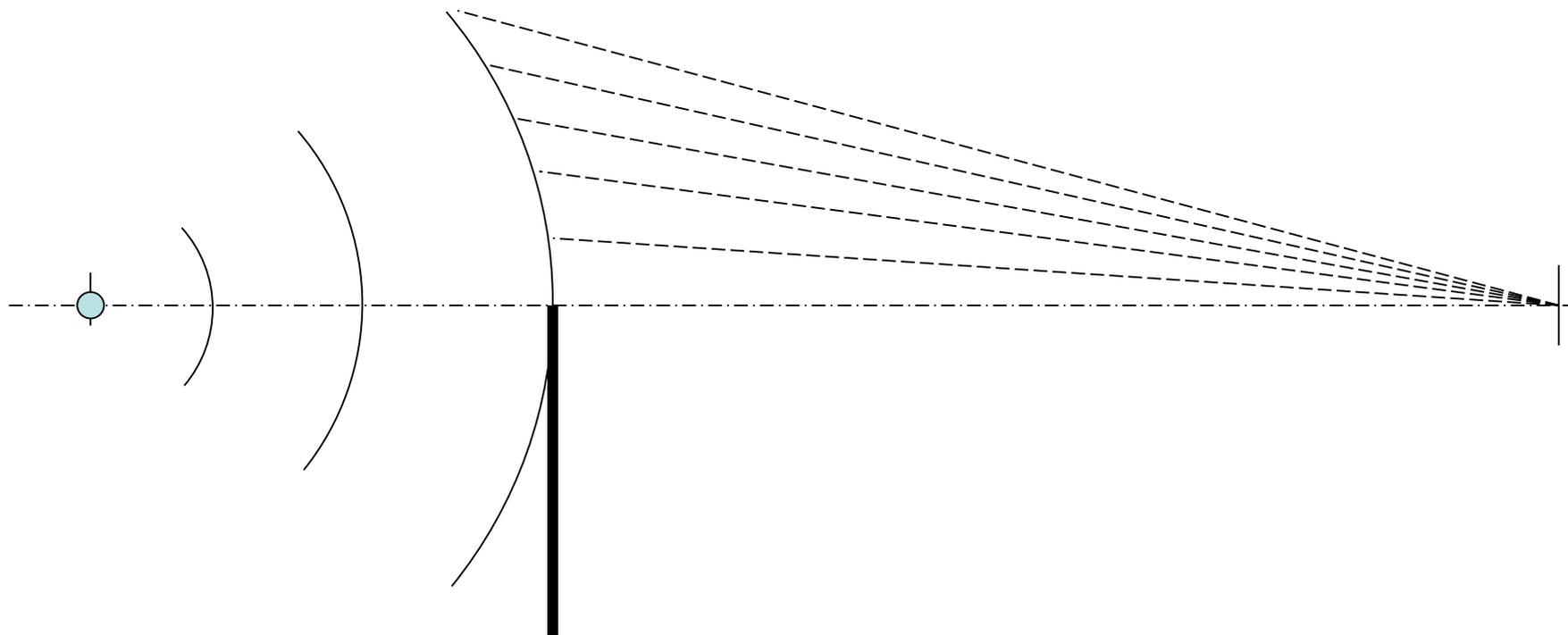
Kinoform透镜

参数：同菲涅尔圆孔



补充B：费涅耳矩孔衍射

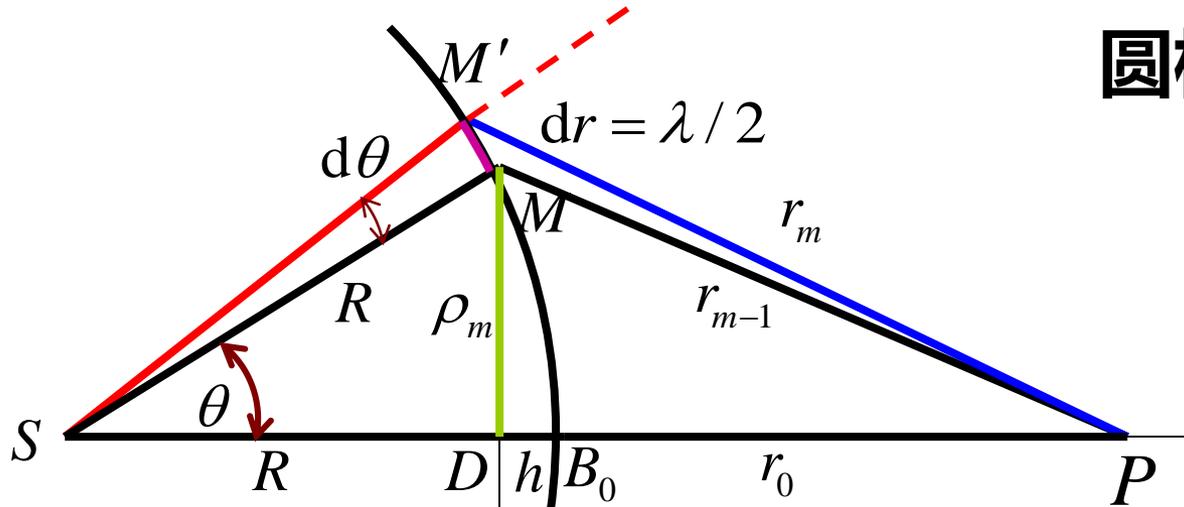
- 光波场中有障碍物，破坏了原有的复振幅分布，导致波场重新分布（波前重构）
- 直边形障碍物，挡住了一半入射光



用半波带方法分析直边衍射

- 首先将露出的波面划分成一系列半波带，用振幅矢量法得到合振动的振幅矢量
- 对每一半波带，细分为等相位差的几部分，由于面积迅速减小而倾角迅速增大，则相邻两部分的振幅相差较大

圆柱形波面面积



$$dS = lRd\theta$$

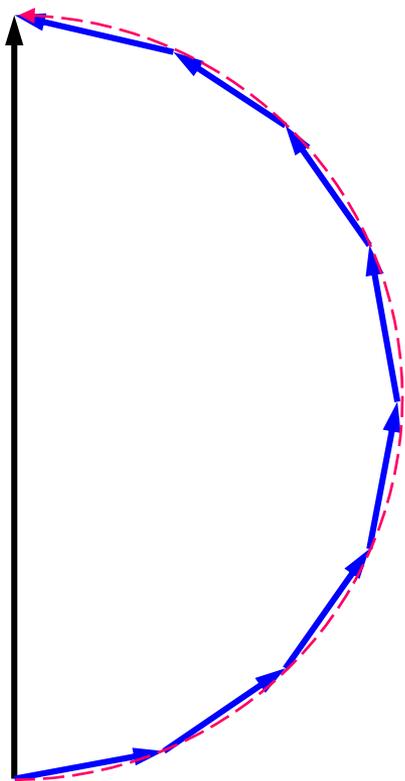
ΔSMP 中 $\cos \theta = \frac{R^2 + (R + r_0)^2 - r_m^2}{2R(R + r_0)}$

$$d\theta = \frac{r_m}{R(R + r_0) \sin \theta} dr_m \quad dS = \frac{l r_m}{(R + r_0) \sin \theta} dr_m$$

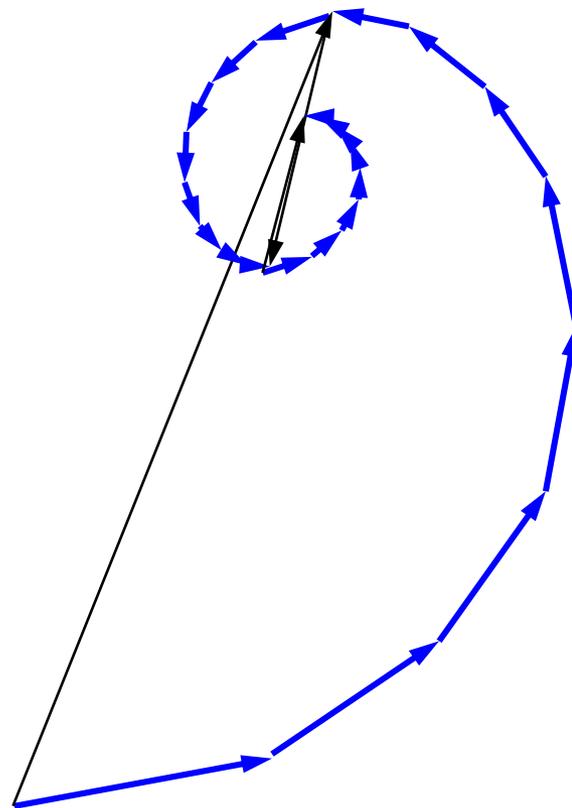
$dr_m = \lambda / 2 \quad dS = S_m$ 第 m 个半波带的面积

$$\frac{S_m}{r_m} = \frac{l\lambda}{R + r_0} \frac{1}{2 \sin \theta_m} \quad \text{随着 } m \text{ 的增大, 复振幅快速减小}$$

直边衍射的强度分析

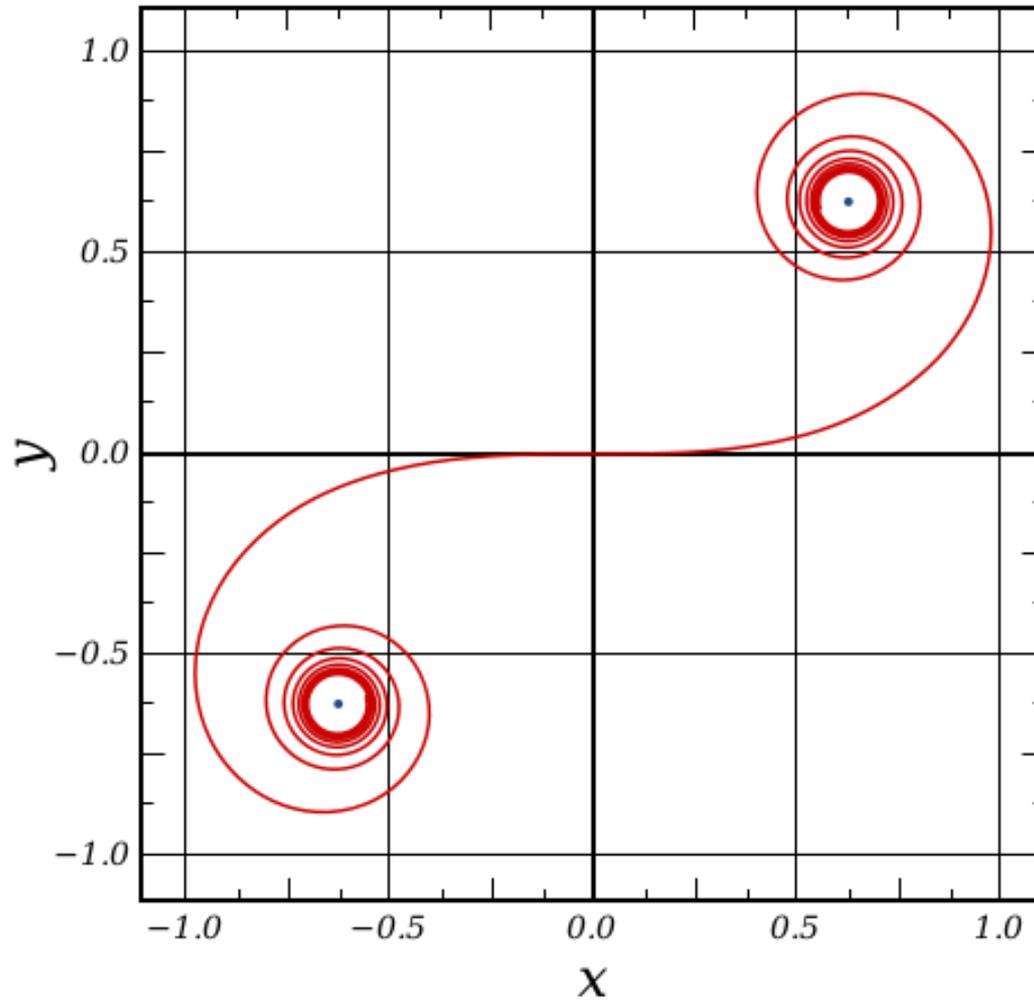


圆孔衍射一个半波带的
振幅矢量

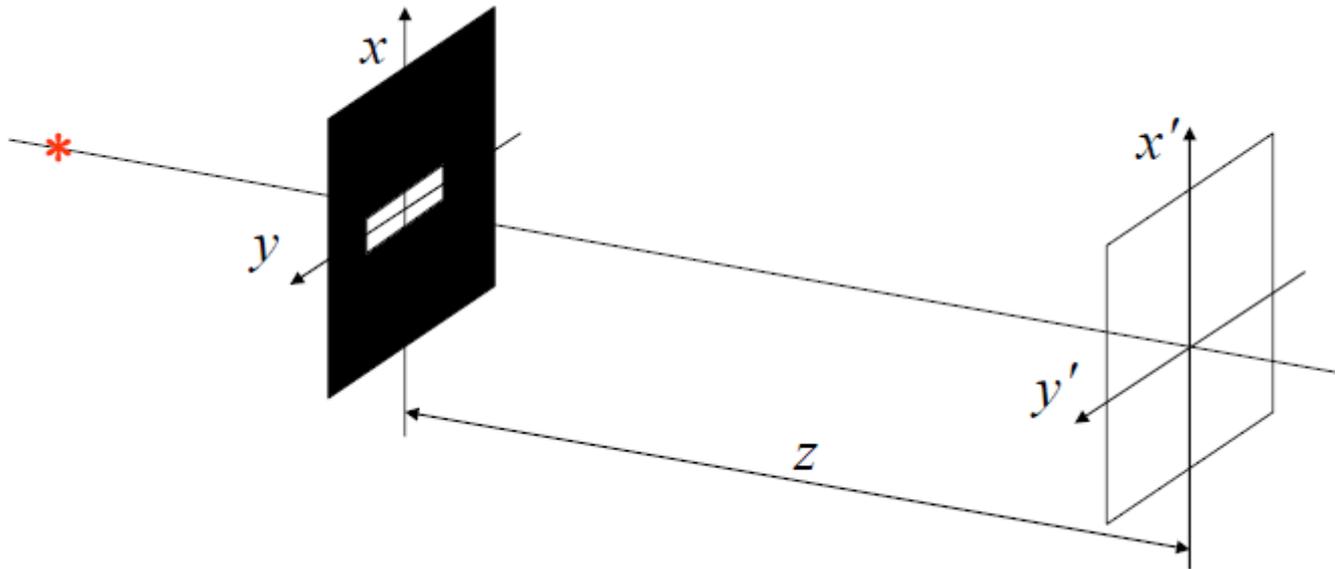


直边衍射一个半波带的
振幅矢量

Cornu spiral



补充B：费涅耳矩孔衍射



$$\tilde{U}(x', y') = c \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} e^{ik \frac{(x-x')^2 + (y-y')^2}{2z}} dx dy$$

$$\tilde{U}(x', y') = c \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} e^{ik \frac{(x-x')^2 + (y-y')^2}{2z}} dx dy$$

作积分变换：

$$\mu = \sqrt{\frac{2}{\lambda z}}(x - x'), \nu = \sqrt{\frac{2}{\lambda z}}(y - y')$$

$$\tilde{U}(x', y') = c \int_{\mu_1}^{\mu_2} \int_{\nu_1}^{\nu_2} e^{i \frac{\pi \mu^2}{2}} d\mu e^{i \frac{\pi \nu^2}{2}} d\nu$$

其中：

$$\mu_1 = -\sqrt{\frac{2}{\lambda z}}\left(\frac{a}{2} + x'\right), \mu_2 = \sqrt{\frac{2}{\lambda z}}\left(\frac{a}{2} - x'\right)$$

$$\nu_1 = -\sqrt{\frac{2}{\lambda z}}\left(\frac{b}{2} + y'\right), \nu_2 = \sqrt{\frac{2}{\lambda z}}\left(\frac{b}{2} - y'\right)$$

费涅耳积分和考纽螺线(Cornu spiral)

$$F(w) = \int_0^w e^{i\frac{\pi t^2}{2}} dt = C(w) + iS(w)$$

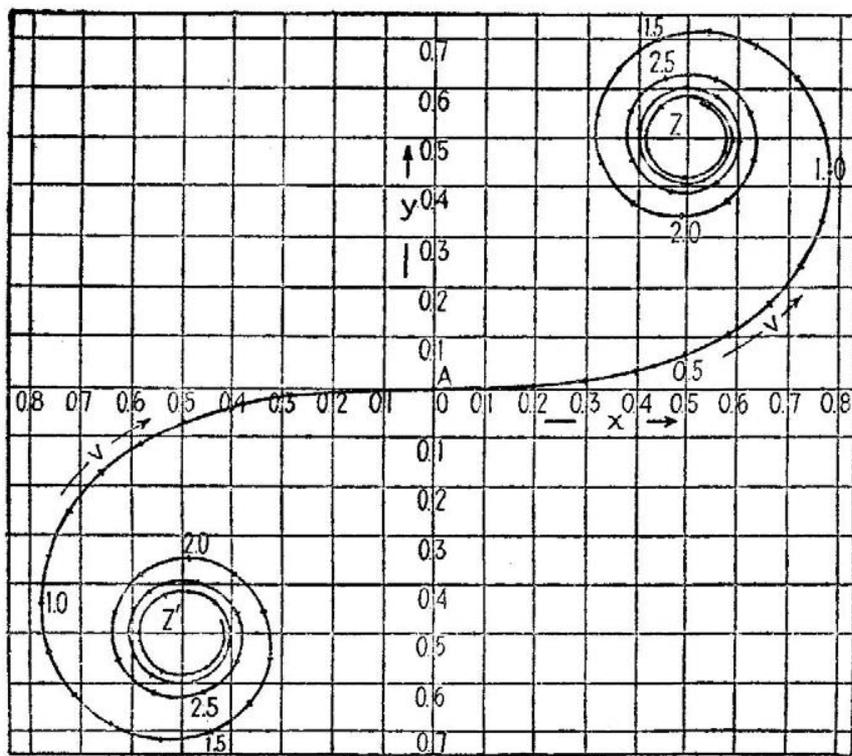
$$\text{其中: } C(w) = \int_0^w \cos\left(\frac{\pi t^2}{2}\right) dt$$

$$S(w) = \int_0^w \sin\left(\frac{\pi t^2}{2}\right) dt$$

分别称为菲涅尔余弦积分和正弦积分

$$F(\pm\infty) = \pm \frac{1}{2}(1 + i)$$

以 $C(w)$ 为横坐标, $S(w)$ 为纵坐标作图:



$$\int_{v_1}^{v_2} e^{i \frac{\pi t^2}{2}} dt$$

$$= \int_0^{v_2} e^{i \frac{\pi t^2}{2}} dt - \int_0^{v_1} e^{i \frac{\pi t^2}{2}} dt$$

$$= F(v_2) - F(v_1)$$

为螺线上一段长 $v_2 - v_1$ 弧,
但弦长取决于:

$$F(v_2) - F(v_1)$$

$$\tilde{U}(x', y') = c [F(\mu_2) - F(\mu_1)] \cdot [F(v_2) - F(v_1)]$$

衍射强度

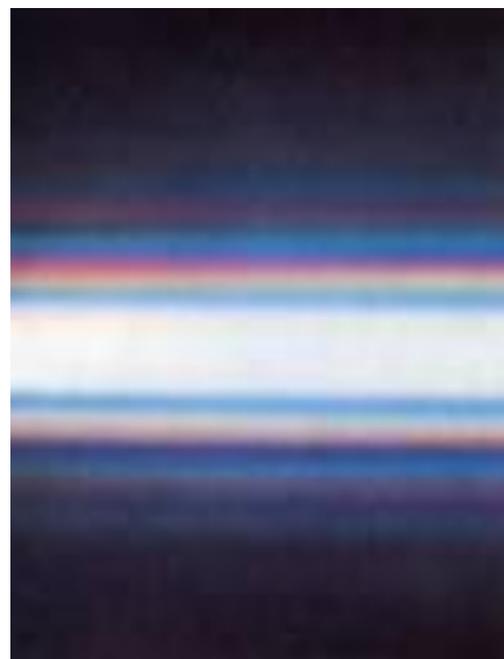
i) 菲涅尔单缝(single slit)衍射

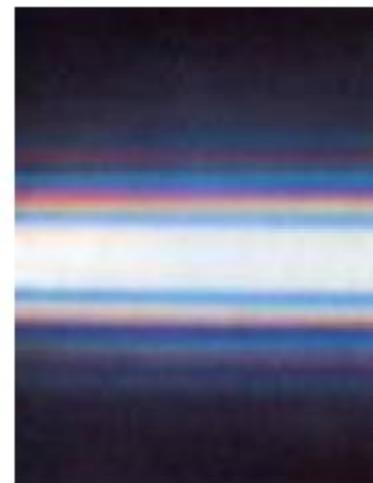
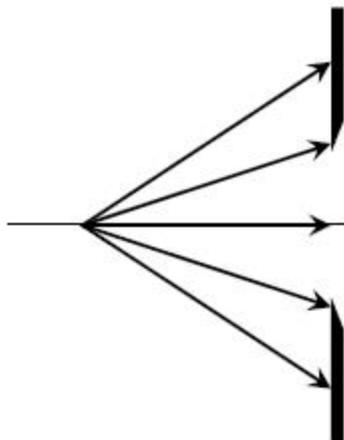
$$\mu_2 = \sqrt{\frac{2}{\lambda z}} \left(\frac{a}{2} - x' \right), \quad \mu_1 = -\sqrt{\frac{2}{\lambda z}} \left(\frac{a}{2} + x' \right)$$

$$v_2 = +\infty, \quad v_1 = -\infty$$

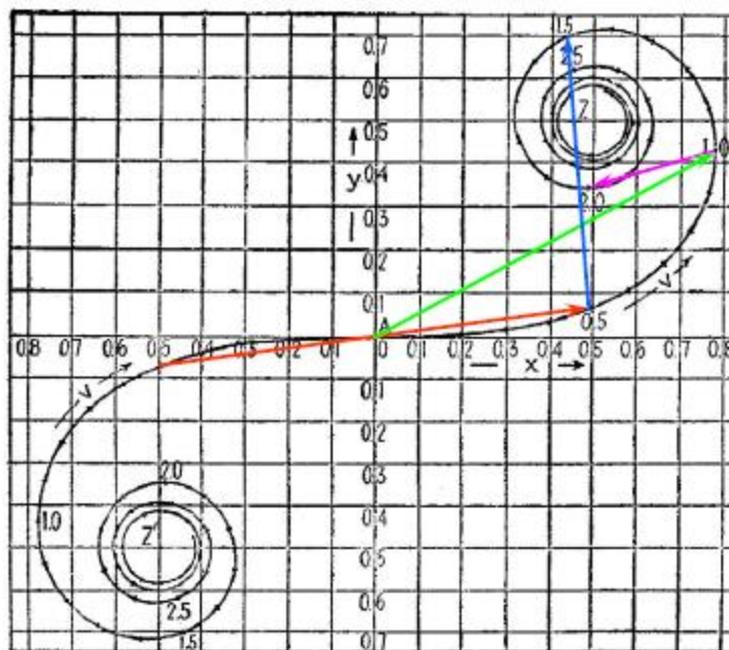
$$\tilde{U}(x') = c [F(\mu_2) - F(\mu_1)]$$

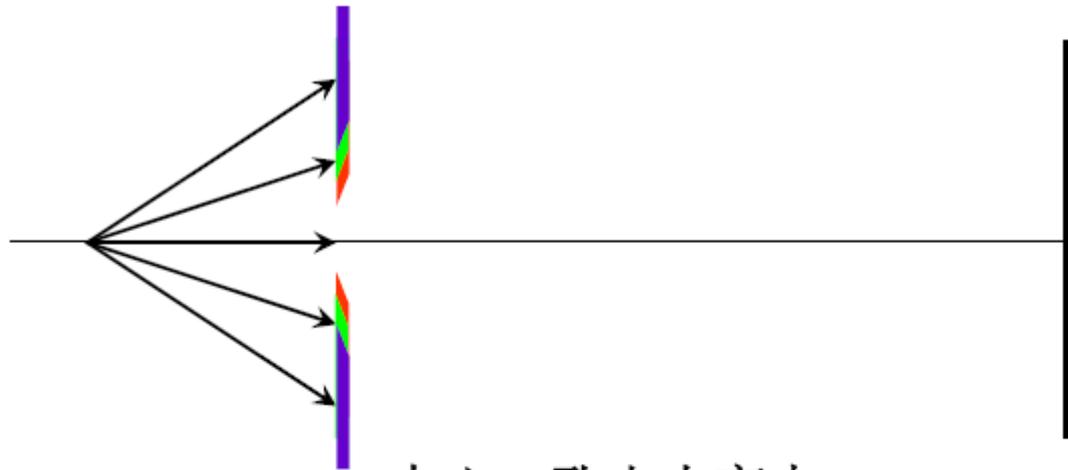
$$I(x') = I_0 |F(\mu_2) - F(\mu_1)|^2$$



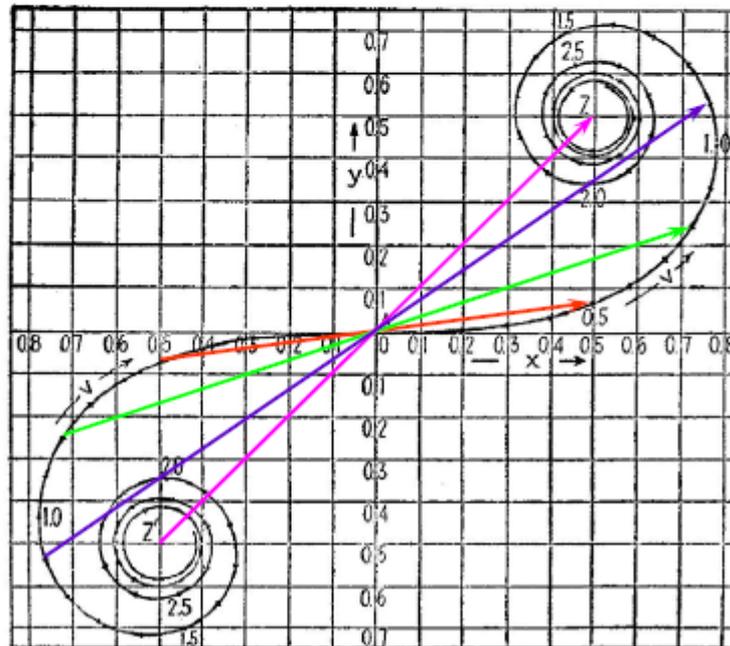


孔由中心向边缘移动





中心，孔由小变大



ii) 菲涅尔直边(single edge)衍射

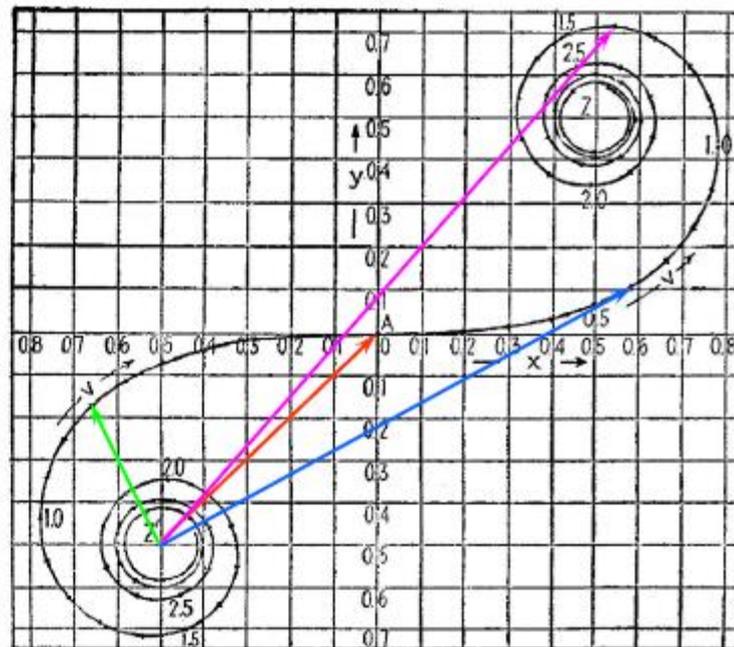
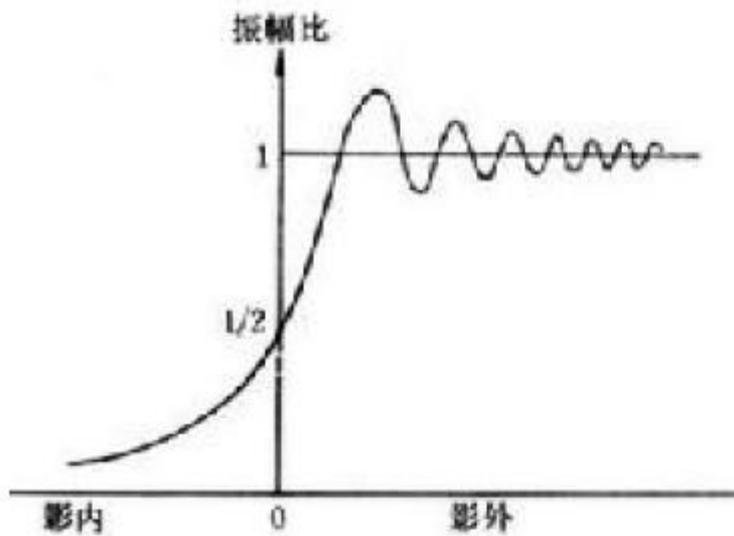
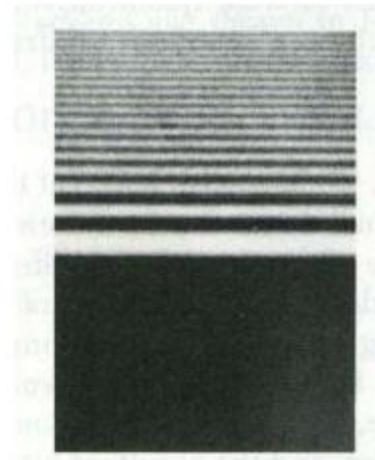
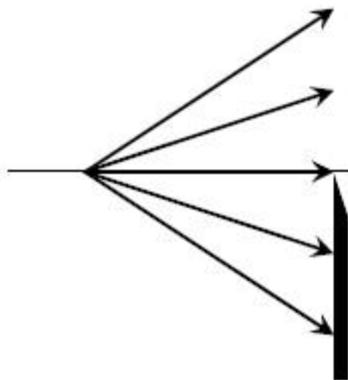
$$\mu_2 = +\infty, \quad \mu_1 = -\sqrt{\frac{2}{\lambda z}}\left(\frac{a}{2} + x'\right)$$

$$v_2 = +\infty, \quad v_1 = -\infty$$

$$\tilde{U}(x') = c \left[\frac{1}{1} (1+i) - F(\mu_1) \right]$$

$$I(x') = I_0 \left| \frac{1}{2} (1+i) - F(\mu_1) \right|^2$$





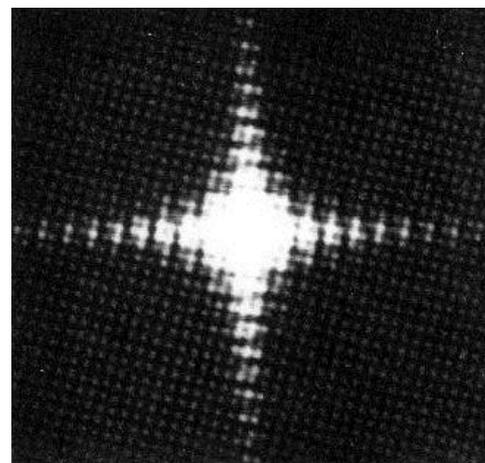
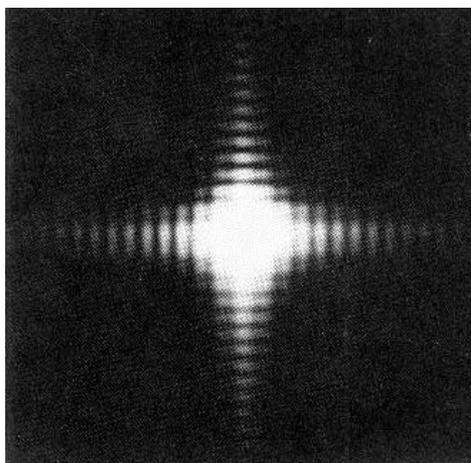
iii) 菲涅尔矩孔(rectangular aperture)衍射

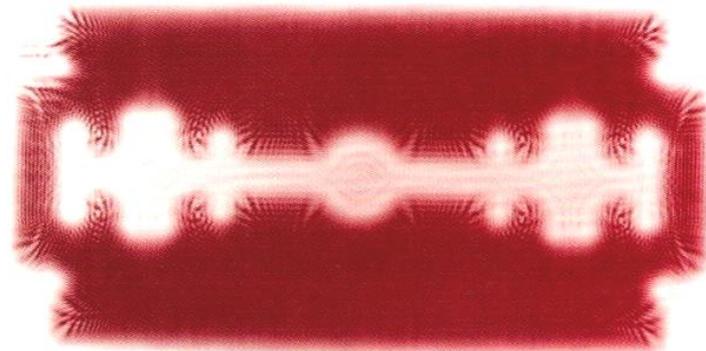
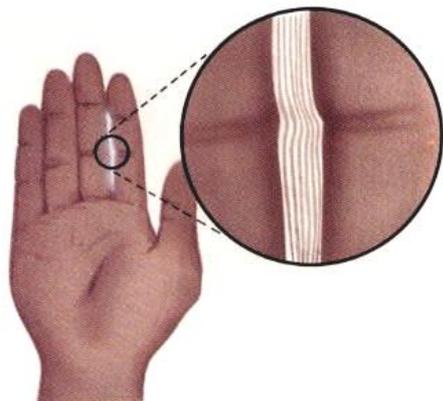
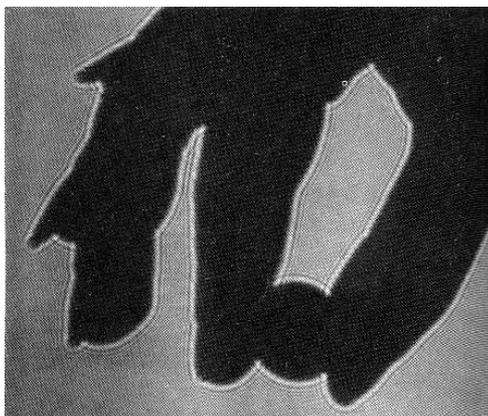
$$\mu_2 = \sqrt{\frac{2}{\lambda z}} \left(\frac{a}{2} - x' \right), \quad \mu_1 = -\sqrt{\frac{2}{\lambda z}} \left(\frac{a}{2} + x' \right)$$

$$v_2 = \sqrt{\frac{2}{\lambda z}} \left(\frac{b}{2} - y' \right), \quad v_1 = -\sqrt{\frac{2}{\lambda z}} \left(\frac{b}{2} + y' \right)$$

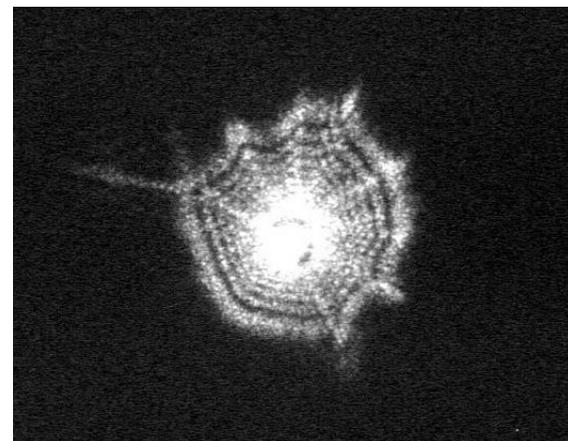
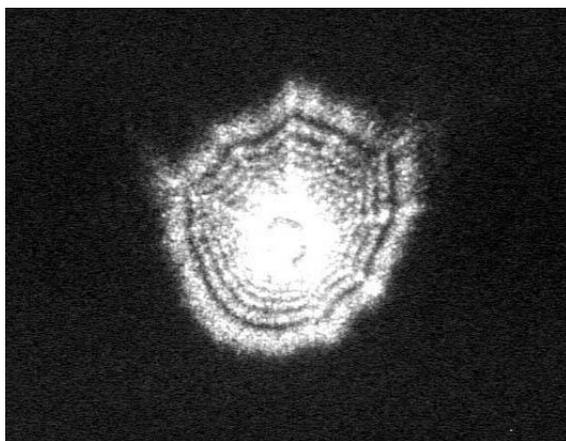
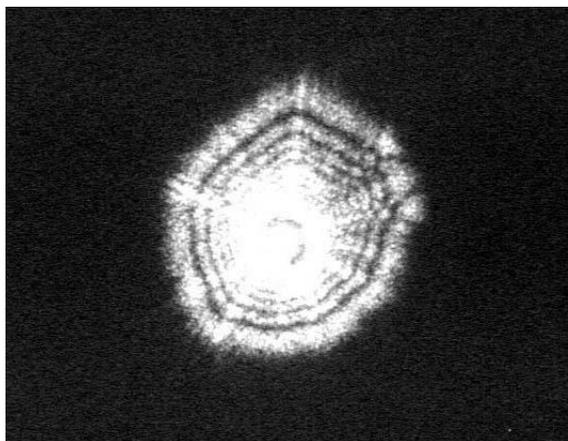
$$\tilde{U}(x', y') = c [F(\mu_2) - F(\mu_1)] \cdot [F(v_2) - F(v_1)]$$

$$I(x', y') = I_0 |F(\mu_2) - F(\mu_1)|^2 |F(v_2) - F(v_1)|^2$$

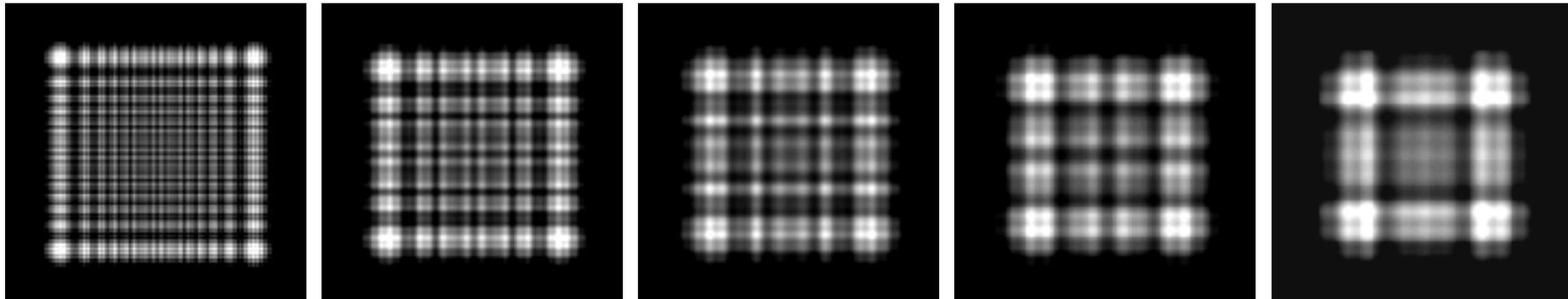




任意屏的菲涅耳衍射图样



任意屏的菲涅耳衍射图样



(a) $z=0.5\text{m}$

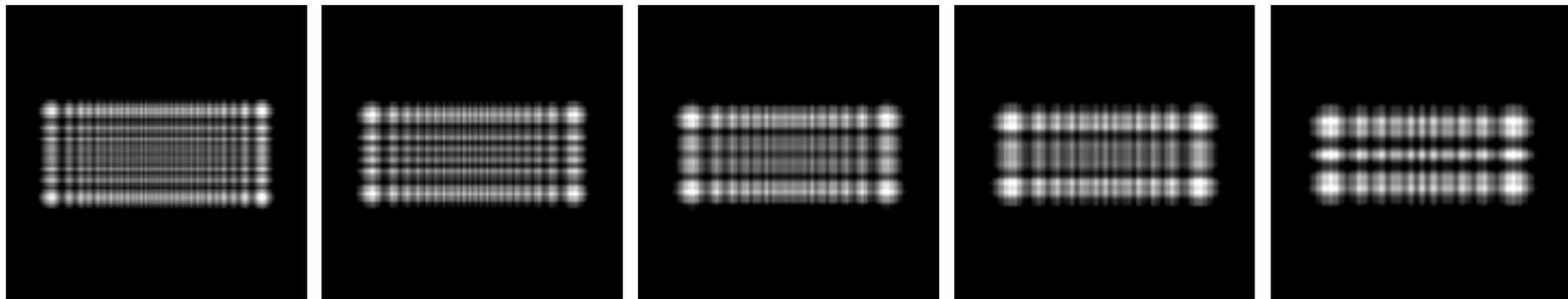
(b) $z=1.0\text{m}$

(c) $z=1.5\text{m}$

(d) $z=2.0\text{m}$

(e) $z=2.5\text{m}$

图4.2-15 正方形孔的菲涅耳衍射仿真图样（不同观察平面上）



(a) $z=1.0\text{m}$

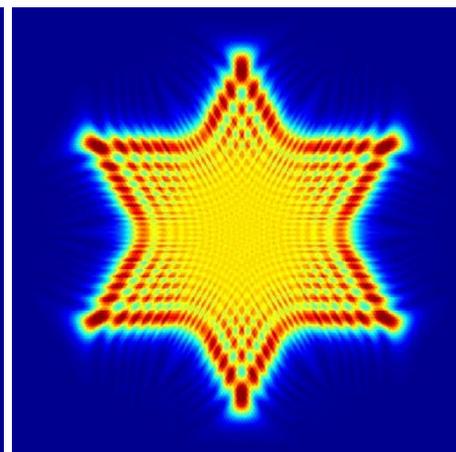
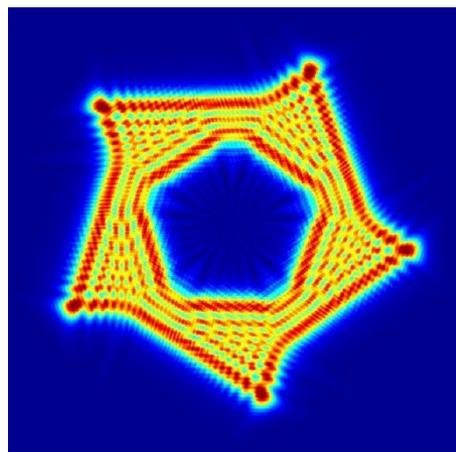
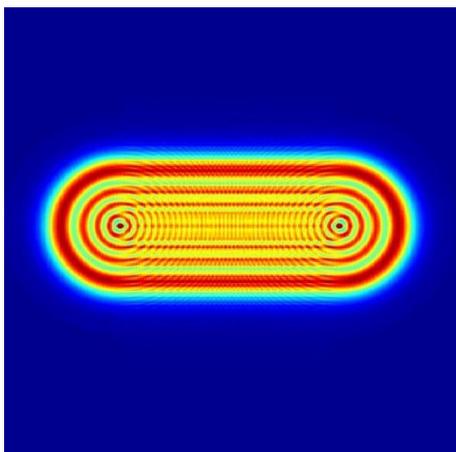
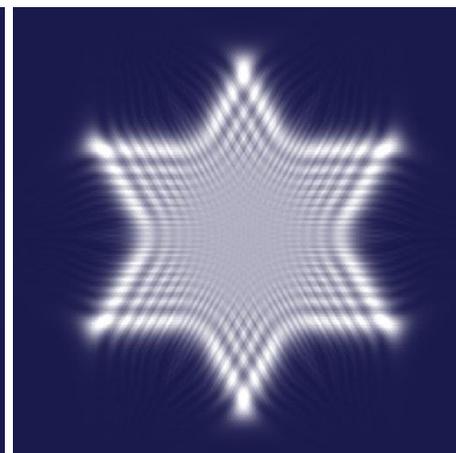
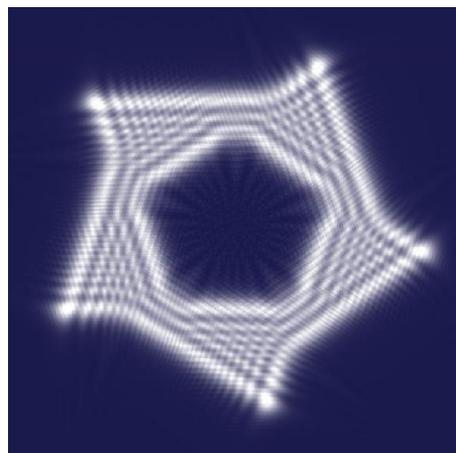
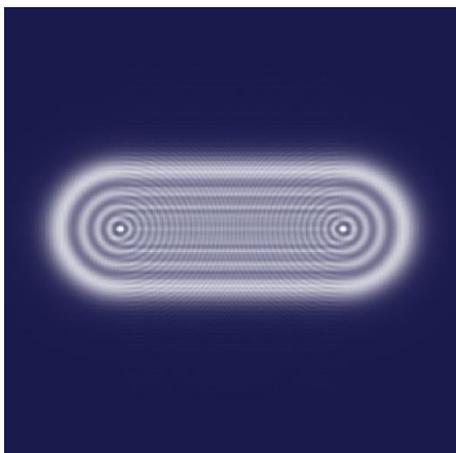
(b) $z=1.5\text{m}$

(c) $z=2.0\text{m}$

(d) $z=2.5\text{m}$

(e) $z=3\text{m}$

矩形孔的菲涅耳衍射仿真图样（不同观察平面上）



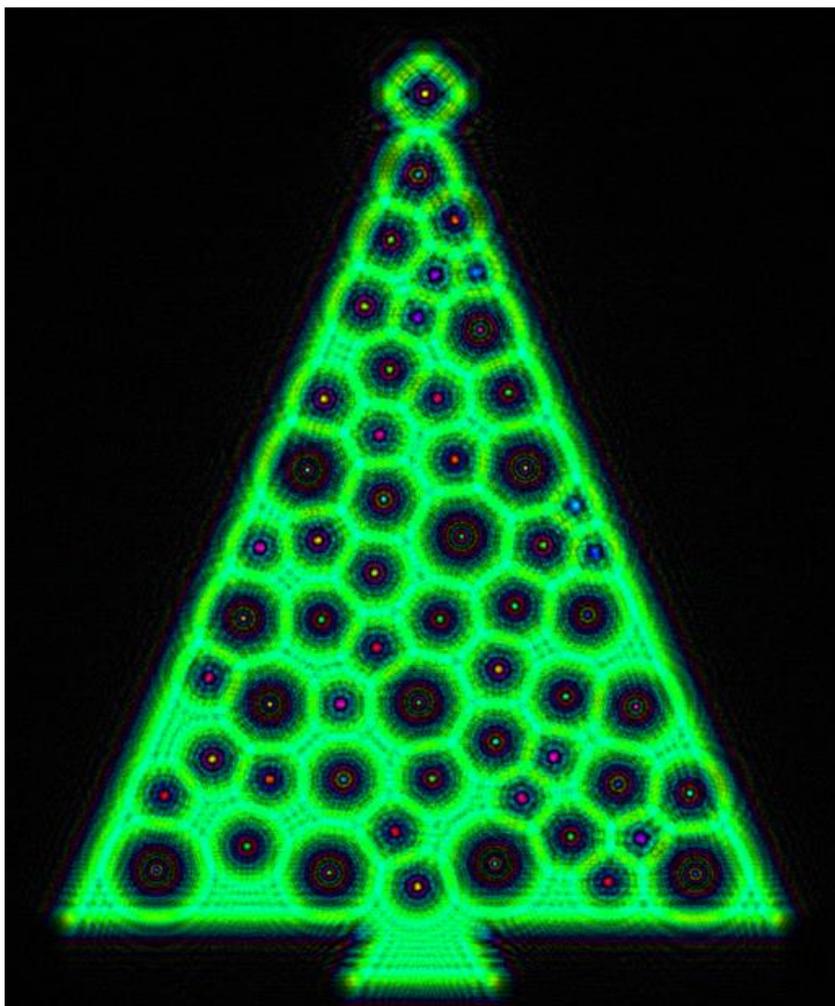
圣诞树

椭圆跑道

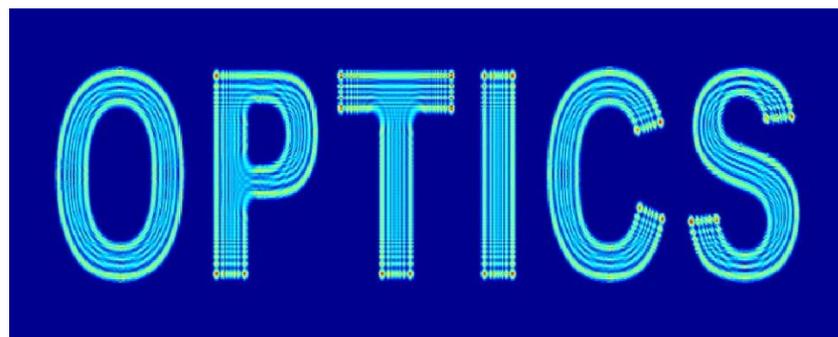
梅花飞镖

六角星

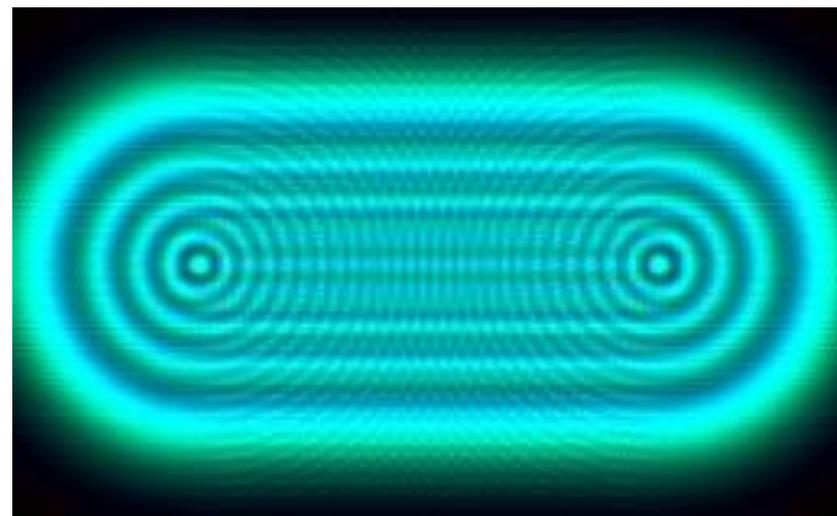
任意屏的非涅耳衍射仿真图样



圣诞树状的菲涅耳衍射图样



字母的菲涅耳衍射图样



跑道形孔的菲涅耳衍射图样